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NOTE ON THE PARALLEL-POSTULATE.

BY D. J. STEPHENS.

A RECENT report on the teaching of geometry by a committee of the Mathematical Association (*The Teaching of Geometry in Schools*) suggests that a coherent geometrical system could be erected on a foundation other than that of Euclid's axioms and postulates. Indeed, a cogent argument is advanced for the replacement of the principle of superposition and the parallel-postulate by the "principles of congruence and similarity" which are regarded as the expressions of more profound beliefs. The success of the recommendation, to substitute the principle of congruence for the principle of superposition, is undoubted; and the comparative reluctance to adopt the twin "principle of similarity" may be due, in part, to an obscurity in the treatment of parallels with this principle as basis.

The report develops the various propositions relating to parallelism by, first, demonstrating that if the exterior angle be equal to the interior opposite angle on the same side of the cutting line, the lines so cut are parallel; thence, by establishing the parallel-postulate which follows from a proposition showing that, if the exterior angle be either greater or less than the interior angle, the lines must meet. But in the proof of the latter proposition there is an assumption which makes the validity of the proof questionable.

Let us examine this proposition. BX and CY are two lines cut by the transversal ABC such that the angle ABX is greater than the angle ACY. Then BX and CY must intersect. Take an arbitrary point S in BX, and join CS. Then the angle ACS is either (1) greater than, (2) equal to, or (3) less than the angle ACY.

In case (1), there is a point H between B and S such that the angle BCH is equal to the angle BCY, and CY cuts BX at H.

In case (2), CY cuts BX at S.

In case (3), since the angle BCY is greater than the angle BCS and less than the angle ABX, there is at least one point K between B and C such that the angle BKS is equal to the angle BCY.

Since the angle XBC is common, the figure $XBCY$ has two angles equal to angles in the triangle BKS and is, therefore, similar to it. Consequently, BX and CY must meet.

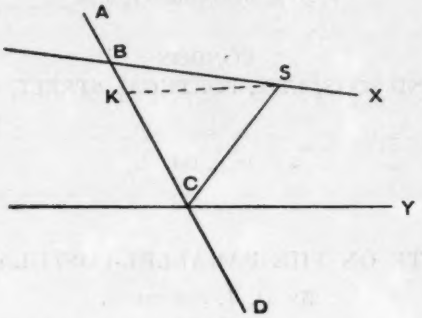

FIG. 1.

But, in the third case, two assumptions are involved:

(1) that a point K exists such that the angle BKS is equal to the angle BCY; and

(2) that if the point K does exist, it lies between B and C.

Of these assumptions the latter can be justified. For if K lies in AB, the angle AKS is greater than the angle ABS, which is impossible; and if K lies in CD, the angle AKS is less than the angle BCY which, again, is impossible. Therefore K, if it exists, cannot lie outside BC.

Let us examine the remaining assumption. Let BC be bisected at R.

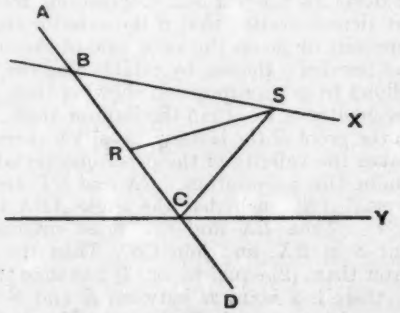

FIG. 2.

Then the angle BRS is either (1) equal to, (2) less than, (3) or greater than the angle BCY.

In case (1) the point K coincides with R, in which case K is existent.

In cases (2) and (3) the point K lies in BR and RC respectively.

By continuing the process of joining the point S to the mid-point of the segment of BC in which the point K, if existent, must lie, an infinitesimally small segment will be obtained eventually; but as the process may be continued *indefinitely* there seems to be no way of establishing the actual existence of K. A point K' can certainly be found such that the angle $BK'S$ differs from the angle BCY by a quantity which may be made as small as we please.

It is evident that a deduction of the assumption involved in the proof in case (3) is required; or the assumption must rank as an additional axiom. I suggest that the following proof is both teachable and sound.

With the data as stated, let a line be drawn at B, BR, such that the angle ABR is equal to the angle BCY. Take any point F in BR. Join F and C. As BR rotates, it traces out, in passing from BR to BD, every angle between the angle ABR and a straight angle; and some point in BR as it rotates will always lie on FC. So that when BR is in the position BX, let the point W be the point which

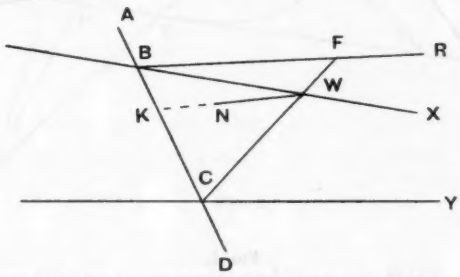

FIG. 3.

lies on both FC and BX. From W draw a line making an angle CWN which shall be equal to the angle WFB. Then $NWCB$ possesses two angles equal to angles in the triangle BCF, and is, therefore, similar to it. Hence, WN meets BC at a point K, and the angle WKC is equal to the angle FBC. The supplementary angle WKA is equal to FBA and, so, BCY. Therefore, WK is a line drawn from a point W in BX at a given angle to $ABCD$.

The proof of the proposition is then straightforward.

Consider the figure $XBCY$ and the triangle WBK; the angle WBK is common, and BKW is equal to BCY.

Therefore the figure $XBCY$ is similar to the triangle WBK, and BX and CY, the sides of the triangle, must meet.

D. J. STEPHENS.

A DESCRIPTIVE PROOF OF A THEOREM FOR CONIC SECTIONS.

By R. J. Lyons.

1. *To prove that a line meeting one given conic in points harmonic to its intersections with another given conic, touches a conic.*

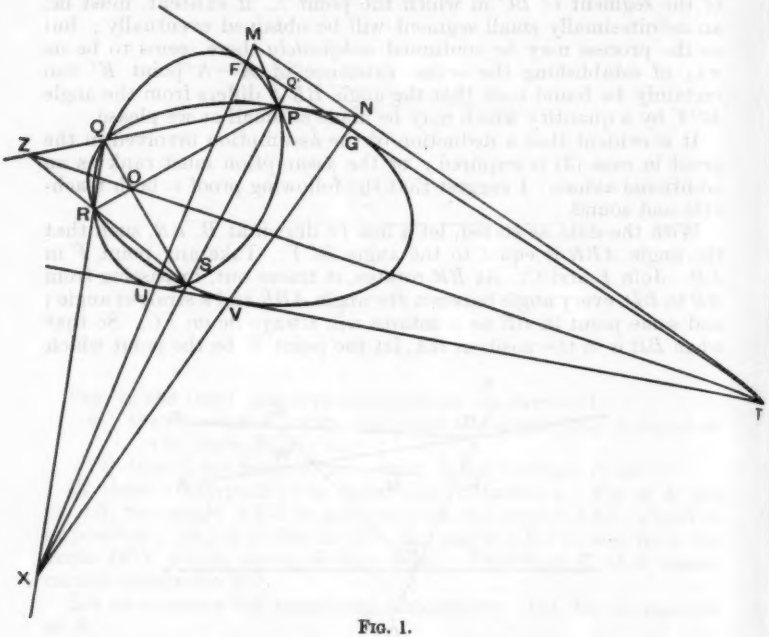

Fig. 1.

2. Lemma I. *Let two conics intersect in P, Q, R, S as shown in Fig. 1.*

Let UV, FG be a pair of common tangents intersecting in T, and let the tangents at P to the two conics cut UF, VG in the points M, N respectively. Then M, N, T are collinear.

For: QR, PS, FU, GV all meet in a point X, and the range $MFUX$ is related to the pencil $P(PFUS)$, which is related to $F(PFUS)$, which is related to $PQ'XS$, where Q' is the intersection of SP and FG. Similarly, $NGVX$ is related to $PQ'XS$. Hence $MFUX$ and $NGVX$ are related, and thus MN passes through T.

3. In Lemma I it has been proved that the tangents at P to the two conics meet XUF and XVG in two points M, N in line with T. We are now able to prove

Lemma II. *PT is the tangent at P to the conic which passes through P, Q, R, U, V.*

Let PSX cut UV in H and MN in K. The pencil $P(USVT)$ is related to the range $UHVT$, and the pencil $P(MSNT)$ is related to the range $MKNT$. Hence, since the ranges $UHVT$ and $MKNT$ are in perspective from X, the pencils $P(USVT)$ and $P(MSNT)$ are related. Therefore, also, the pencils $P(USVT)$ and $P(NTMS)$ are related. Hence P is the vertex of a pencil in involution of which PN, PU ; PM, PV ; PT, PS are conjugate rays.

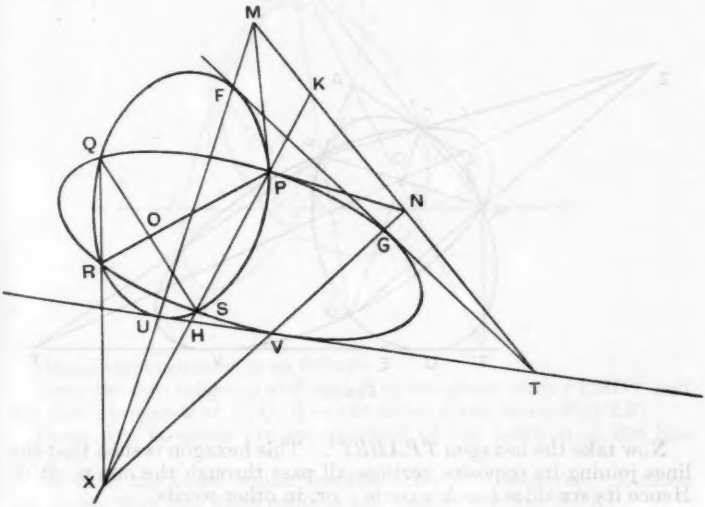

FIG. 2.

Now since the two given conics and the conic $PQRUV$ (not drawn in the diagram) have three points in common—P, Q, R—and U, V are respectively the intersections of the two given conics with the conic $PQRUV$, while S is an intersection of the given conics, the above is a sufficient condition for PT to be the required tangent.*

Let PQ, RS cut in Z ; PS, QR in X ; PR, QS in O. Since T is the intersection of a pair of common tangents to the two given conics, T lies on ZO.†

Hence from Lemma II we can say that the tangent at P to the conic $PQRUV$ is the line joining P to the intersection of ZO and UV. Similarly, if XO cut UV in T', the tangent at R to the conic $PQRUV$ is RT'.

* See Baker, *Principles of Geometry*, ii. 61, par. 2.

† Baker, ii. 37.

4. Lemma III. Consider now the conic passing through P, Q, R, S and an arbitrary point E on UV. This conic cuts UV in a second point L, which is the harmonic conjugate of E for U and V. Let the tangent at Q to this conic $PQRSE$ cut OX in A and OZ in B; then AP and BR are also tangents to the conic $PQRSE$.*

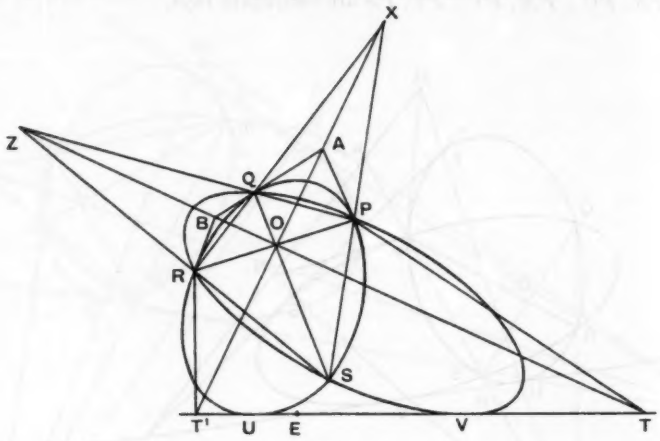

FIG. 3.

Now take the hexagon $TPABRT'$. This hexagon is such that the lines joining its opposite vertices all pass through the one point O. Hence its six sides touch a conic; or, in other words,

UV is a tangent to the conic which touches the five lines TP, PA, AB, BR, RT'.

5. *The Theorem.* Given two arbitrary conics $PQRUV$ and $PQRLE$, let $LUEV$ be a line such that its intersections L, E with one of these conics are harmonic conjugates of one another with respect to U, V, its intersections with the other conic. It is required to prove that the line $LUEV$ touches a fixed conic.

Let the points of intersection of these two conics be P, Q, R, D. Of these four intersections take any three, say P, Q, R. Draw two conics—shown in the diagram by dotted lines—to pass through P, Q, R and to touch the line $LUEV$ at U and V respectively. Let these two latter conics cut at S; then S lies upon the conic $PQRLE$. (To prove this statement we have only to draw the conic $PQRSE$. This conic must pass through the harmonic conjugate of E with respect to U and V; that is, $PQRSEL$ all lie on the same conic.)

To these two conics given by the dotted lines we can apply the

* Baker, ii. 37.

lemmas we have established. In the notation of Lemma III,

TP is the tangent at *P* to the conic *PQRUV* ;
PA is the tangent at *P* to the conic *PQRLE* ;
AB is the tangent at *Q* to the conic *PQRLE* ;
BR is the tangent at *R* to the conic *PQRLE* ;
RT′ is the tangent at *R* to the conic *PQRUV*.

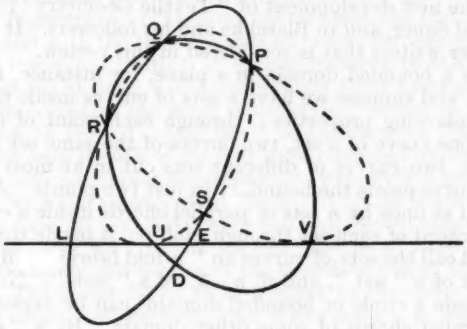

FIG. 4.

Hence the argument is as follows :

Draw the two tangents at *P* and *R* to the given conic *PQRUV* and the three tangents at *P*, *Q*, *R* to the other given conic *PQRLE*.

These five tangents are independent of the position of the line *LUEV*.

Describe the conic which touches these five tangents. Lemma III shows that this fixed conic is touched by the line *LUEV*.

R. J. LYONS.

GLEANINGS FAR AND NEAR.

859. From *Conway Letters*, by M. H. Nicholson (Yale University Press, 1930), No. 85 (p. 146). Henry More to Anne Conway, March, 1658.

" I hear by Mr. Hyrne what a quick Proficient your Ladyship is already at Mathematicks. . ."

P. 146, f.n. John Finch to Anne Conway. December 3/13, 1657.

" For the study of the Mathematiques may take up all ones life . . . but I would not have you study Mathematiques for any other end, then to make you capable of anything in Astronomy or Optics or morall Philosophy, which may be done by the Helpe of the first 6 Bookes of Euclid."

No. 151 (p. 231). Henry More to Anne Conway. October 17, 1664.

" I have renew'd my acquaintance with such Mathematicall Theorems as I was in some measure conversant in before I fell a Theologizing, which was so long, that I had almost forgott all that little I knew in Geometry which I have recovered with some advantage, and my health, I hope, into the bargain." [Per Mr. F. Puryer White.]

TEXTILE GEOMETRY.

By H. G. Forder.

1. For a subject whose death has been several times announced, Geometry is at present in rather a thriving state. During this century considerable progress has been made in two kinds of topology, in general differential geometry and in the theory of ovals, and now we have the new development of "Textile Geometry", largely due to Graf and Sauer, and to Blaschke and his followers. It is the work of the latter writers that is considered in this review.

Consider a bounded domain in a plane, for instance, the interior of a circle, and suppose we have n sets of curves inside the domain, with the following properties: through each point of the domain goes just one curve of a set, two curves of the same set never meet each other, two curves of different sets cut in at most one point, and each curve meets the boundary in just two points. An example is provided at once by n sets of parallel chords inside a circle, when there is a chord of each set through each point inside the circle.

We shall call the sets of curves an "n-fold fabric"; if $n=2$, we shall speak of a "net", and if $n=3$, of a "web". Now a net of curves (inside a circle or bounded domain) can be depicted on two sets of parallel chords of some other domain. By a "depiction" we shall mean a one-to-one correspondence which is continuous in both senses, *i.e.* continuous motion in the original corresponds to continuous motion in the picture, and *vice versa*. We can put our statement popularly by saying that a net of curves in a domain can be continuously deformed until both sets of curves are straightened out into two sets of parallel lines.

For a web, however, matters are different; it is not always possible to straighten out the three sets of curves into three sets of parallel lines. For this to be possible the curves must satisfy a condition which we now give.

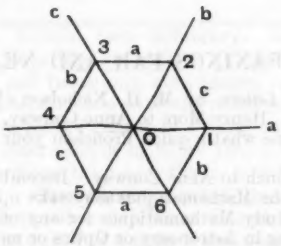

Fig. 1.

Denote the sets of curves by a, b, c; through any point O draw the curves of each set; on the a-curve take a point 1, and through 1 draw the c-curve to meet the b-curve through O in the point 2; through 2 draw the a-curve to meet the c-curve through O in the point 3; continue this process as indicated in Fig. 1. The necessary

and sufficient condition in question is that after one circuit we again reach the point 1. If this condition is satisfied the web is called a " hexagonal web ", and it can, as we said, be depicted on three sets of parallel chords. The simplest such web is that formed of three sets of parallel lines inclined at angles of 60° to one another, and every hexagonal web can be depicted on this simplest case (or on a portion of it). The most general hexagonal web which is made up of straight lines, is the set of tangents to a curve of class three, general or degenerate. (This gives contact with the theory of elliptic functions and with the theorem that all cubics through eight general points go through a ninth.) A special instance of this is the case of three pencils of lines, which accordingly form a hexagonal web.

Blaschke considers the most remarkable theorem in this stretch of the work to be : *A 4-fold fabric is depictable on four pencils of lines if and only if each of the four webs contained in the fabric is a hexagonal web.* A similar theorem holds whenever n exceeds 5, but for $n=5$ we have an exceptional case. A 5-fold fabric whose contained webs are all hexagonal webs can be depicted *either* on a fabric of five pencils of lines *or else* on four pencils of lines no three of whose vertices are collinear, and the pencils of conics through these vertices. The latter fabric is equivalent to the fabric of the five pencils of conics each through four of five general points.

As illustrating old projective theory, we have the result that a depiction of the projective plane which turns four pencils of lines into such is projective; and as illustrating differential geometry, the only surfaces with hexagonal webs made up of geodesics are surfaces of constant curvature.

We have seen what is the condition that a 4-fold fabric can be depicted on four pencils of lines. When can it be depicted on four pencils of parallel lines ? The condition is that the Desargues' figure (Fig. 2) should hold in two ways. In Fig. 2 the a-curves play a

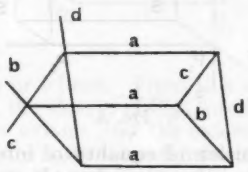

FIG. 2.

special part ; if we assume the existence (*i.e.* the closure) of all such figures, and of the figures when the b-curves play this special part, then will follow the existence of the figures wherein the c and d-curves play this part. Thus if the figure holds in two ways, it holds in all four, and this is the necessary and sufficient condition sought.

2. When the subject had been developed to this stage, it was natural to ask if it threw any light on the foundations of Geometry, for the importance of figures that were parts of fabrics had been shown by Hilbert in his work on Desargues' and Pappus' theorems.

Reidemeister,* who has made important contributions to the theory of fabrics, has now applied this theory to the foundations of *plane* geometry in the book under review. Beginning with a treatment, extending to 60 pages, of the axioms of algebra and of algebraic affine geometry in the absence of the commutative law of multiplication, he deals in the rest of his book mainly with affine plane geometry ; the notion of parallels is thus prominent throughout. Every schoolboy knows that the opposite sides of a parallelogram are equal ; how far can we use this to define equality of length and to set up a co-ordinate system ?

The author considers first a web of " points " and " lines " (point and line being undefined), and he calls the lines of each set in the web " parallel " ; the fundamental properties of parallels are assumed, *e.g.* if $p \parallel q$ and $q \parallel r$ then $p \parallel r$, through each point goes just one parallel to a line of each set (angles are of course outside the scope of the enquiry). Thus essentially we start with a web of " curves ". The problem is to postulate such properties of this web that the curves have all the properties of straight lines, in the affine plane, and that other straight lines, on the plane and not in the web, can be introduced.

As a start we may define the equality of certain vectors ; two intervals P_1P_2 and Q_1Q_2 on distinct lines of one of the three sets (say the a-set) are said to be " equal " when *either* P_1Q_1 and P_2Q_2 fix lines of the same one of the other two sets, *or* (if this is not the case), when there is an interval R_1R_2 such that P_1R_1, P_2R_2 fix lines in the same one of the other two sets, and Q_1R_1, Q_2R_2 fix lines in the other one of these two sets (see Fig. A). The latter alternative

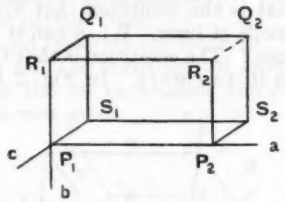

FIG. A.

also serves as a definition of equality of intervals along the same a-line. It will be noted that the intervals are like " directed intervals " or " vectors " of ordinary geometry.

If the equality-relation of this definition is to be transitive, then it is necessary and sufficient that all such figures as Fig. A in the plane should close ; that is, if we know that P_1P_2, Q_1Q_2, R_1R_2, S_1S_2 are a-lines and P_1R_1, P_2R_2, S_1Q_1, S_2Q_2 are b-lines and P_1S_1, P_2S_2 Q_1R_1 are c-lines, then Q_2R_2 is a c-line. (The c-lines here play a special part ; the theorems corresponding for the a and b-lines follow from this one.)

* *Grundlagen der Geometrie.* By K. Reidemeister. Pp. x, 147. Rm. 11.34. 1930. (Springer.)

It is now possible to define the addition of vectors along lines of the *same* set. If we wish this addition to be commutative, we must assume Fig. B closes. It is an exceedingly interesting fact that the

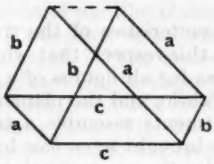

FIG. B.

closure of Fig. B implies that of Fig. A (using only the fundamental properties of parallels), but that the converse is not true.

The figure of the hexagonal web (Fig. 1) is a special case of both Fig. A and Fig. B ; and if we introduce some order axioms, Archimedes' Axiom and a continuity axiom into our web, we can deduce the closure of Fig. B from that of the hexagonal web, *and then set up a one-to-one correspondence between the points of our web and the points of the real (affine) plane.* This is evidently the axiomatic version of the condition that a web of curves can be depicted on three sets of parallel lines.

Now drop, for a moment, the order and continuity axioms, but introduce another set of parallel lines, the *d*-lines, and assume Desargues' theorem (or rather the special case of Fig. C) when P_1Q_1,

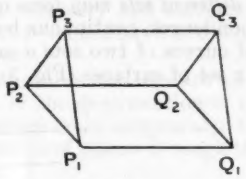

FIG. C.

P_2Q_2, P_3Q_3 are *a*-lines or *b*-lines. From this and the fundamental properties of parallels, the closure of Fig. A follows ; and if we then again adjoin the order axioms and the axiom of Archimedes in a suitable form, we get once again the real (affine) plane, the remaining lines of the plane, not in the web, being suitably defined.

The analysis of the assumptions is pursued in much more detail than we have indicated here, and interesting relations are shown to hold between weakened assumptions and peculiar algebras. The intimate connection between Pappus' theorem and the commutative law of multiplication has long been known, and the connection between Fig. B and the commutative law of addition is not very deep, but Reidemeister considers in full detail the connection between certain figures and the two distributive laws of multiplication. This raises an important unsolved question in universal algebra, whether there are number-systems with multiplication distributive on one

side only, and with non-commutative addition. It may seem strange that this is not known, but while non-commutative multiplication is quite ancient, non-commutative addition is only just being seriously considered.

Summing up the characteristics of the treatment under review, it differs from others in this respect, that whereas they, as a general rule, postulate properties for all figures of a certain type, this does so only for figures in a fabric, and the plane is constructed from the fabric. The other treatments resemble a treatment of algebra in which real numbers are brought in *en bloc* by means of appropriate postulates; this resembles the more usual treatment of algebra in which real numbers are generated from natural numbers. Reidemeister's method is thus that associated with algebra rather than with geometry.

The book deals only with the plane, and the facts on webs mentioned in the first part of this report have to be gathered by reading between the lines; it is a book on the foundations of geometry and not on textile geometry.

3. As the Hamburg school have also pursued researches on space fabrics, we give a résumé of some of the results obtained. (The last paper to hand is numbered 32.) An n-fabric of curves in space is defined inside a sphere, very much as in the case of the plane: through each point must go one curve of each set, and each curve must meet the sphere in just two points, no two curves of the same set may cut, no two of different sets may cut in more than one point, *and no curves of three different sets may form a triangle.* Depictions are one-to-one correspondences, continuous both ways, as before.

If a quadrilateral of curves of two sets a and b always closes we say the sets " span " a set of surfaces (Fig. 3); each surface of the

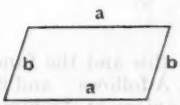

FIG. 3.

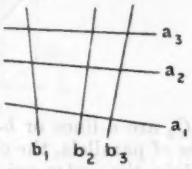

FIG. 4.

set is covered by a net of curves a and b. Then if $n \neq 6$ and each pair of sets of the n-fabric span surfaces, the fabric can be depicted on n bundles of lines (the vertices of the bundles being always outside the domains considered); if $n \leqq 4$, it can be depicted on parallel bundles; but for $n = 6$ there is an interesting exceptional case: the curves are *either* depictable on six bundles of lines *or else* on the following figure: six sets of lines each cutting (outside our domain) a pair of lines; we can name these lines $a_1, a_2, a_3, b_1, b_2, b_3$, so that any two a are skew lines, any two b are skew, but each a cuts each b (Fig. 4). The 6-fabric is then formed of lines cutting the six pairs $a_i a_j$ and $b_i b_j$.

As illustrating old projective theory we have : a depiction which turns six bundles of lines, no three of whose vertices are collinear, into six such, is a projective transformation.

4. The depiction of a set of parallel planes inside a sphere we call a " set " of surfaces. *Three* such sets form a " net ", *four* such a " web ", n such an " n-braid ". The necessary and sufficient condition that a web of surfaces can be depicted on 4 sets of parallel planes is that all octahedra formed of surfaces of the web (so far as they lie in the domain) close (Fig. 5). Call such webs " octahedral webs " ; they correspond to hexagonal webs in the plane.

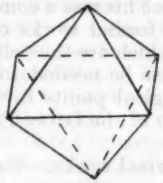

FIG. 5.

If we distinguish the families of surfaces in a web by the letters a, b, c, d, and if the cuts of surfaces of sets a and b, and the cuts of sets c and d are two sets of curves which span surfaces, the surfaces so spanned are called " diagonal surfaces " of the web. Inspection of Fig. 5 shows that, for octahedral webs, there are three sets of diagonal surfaces. A necessary and sufficient condition for a web to be octahedral is that it should have two sets of diagonal surfaces ; it then has a third set.

The octahedral webs whose surfaces are all planes, touch a pencil of quadric envelopes, or the degeneration of such a pencil ; that is, they touch a surface which is the reciprocal of the cut of two quadrics. An example is four pencils of planes through the four sides of a plane or skew quadrilateral.

If a 5-braid is such that each contained web is octahedral, the braid can be depicted on five sets of parallel planes.

The curves in which a surface in a web is cut by the surfaces of the other sets have also been studied. If the webs on the surfaces of three of the sets are hexagonal, this is so for the fourth set of surfaces.

It should be mentioned that some of the above results have only been established under certain differentiability conditions, but that this restriction is being removed. Enough has been said to show that an important new movement in geometry has been inaugurated, connected with subjects as remote as nomography, Einstein's teleparallelism, and Lie's and Brouwer's work on continuous groups. The papers dealing with it are to be found in recent numbers of the *Mathematische Zeitschrift* and the *Hamburg Abhandlungen*, and there are reports, which have been used in the above, by Blaschke in the *Jahresbericht der Deutschen Mathematiker Vereinigung* for 1930 and 1931.

H. G. F.

LEWIS CARROLL AS LOGICIAN.*

By R. B. Braithwaite, M.A.

Whereas the works on Trigonometry, Geometry and Determinants were published over the name of Charles L. Dodgson, Student and Mathematical Lecturer of Christ Church, Oxford, the two small works on Logic—*The Game of Logic* and *Symbolic Logic, Part I*—and the two interesting logical notes in *Mind* appeared over the signature of the creator of Alice. Thus on the theory of double personality which the Rev. C. L. Dodgson tried so hard to establish (and so unsuccessfully, since his was a completely integrated though singular personality), the formal works on logic are to be classed with those informal ones which are the delight of the civilised world. This is my excuse, if excuse be needed, for including in this article some discussion of the logical points involved in the solipsism of the Red King, the monism of the Other Professor and the nominalism of Humpty Dumpty.

To deal first with the formal works. Carroll regarded formal and symbolic logic not as a corpus of systematic knowledge about valid thought nor yet as an art for teaching a person to think correctly, but as a game. He hopes that it may " be of *real* service to the young, and be taken up, in High Schools and in private families, as a valuable addition to their stock of healthful mental recreations ". It has the incidental advantages, unlike " Back-gammon, Chess, and the new Game 'Halma' ", of giving " clearness of thought—the ability to *see your way* through a puzzle . . . and, more valuable than all, the power to detect *fallacies* " (which are encountered " even in sermons ") ; but essentially logic is what Jowett asserted it to be—a dodge—and it was Carroll's aim to make it as good a dodge as possible. With this purpose he produced in 1886 his " Game of Logic " to be played with a board and nine counters, and then developed it into a book of 200 pages published in 1896 with the title of *Symbolic Logic: Part I, Elementary.* He died in 1898 before the projected second and third parts were finished.

Carroll's dodge was to interpret all propositions in terms of the existence or non-existence of classes, and then to represent the classes by regions in spatial diagrams and their existence or non-existence by putting counters or marks in the appropriate regions. The representation of classes by spatial diagrams was introduced by Euler in 1761, and Euler's method represented each class-relationship by a different diagram. It was a definite improvement to use (as Carroll uses) one general schema for all the class-relationships and to state our precise information about the classes by suitably marking the schema. Carroll, however, cannot claim priority in this : it was first proposed by Venn in 1880 and 1883 ; and Carroll's innovation is confined to his using square diagrams which permit of an extension to a greater number of classes than do Venn's circles and ellipses. The symbolic language which he uses for the translation of his diagrams has the virtue of showing

* Charles Lutwidge Dodgson, 1832-1898.

clearly that the propositions are being interpreted in terms of class-relationships, but otherwise it is no improvement on more ordinary statements. It does not show in itself why the conclusion of a syllogism follows from the premisses, but only states that it does.

To illustrate Carroll's method I will take one of his examples (for one of the best features of the book is the examples he invented). Consider what conclusion can be drawn from the two premisses :

> Canaries, that do not sing loud, are unhappy ;
> No well-fed canaries fail to sing loud.

The first premiss asserts the existence of a class of non-loud-singing canaries, and the non-existence of a class of happy non-loud-singing canaries. The second premiss asserts the non-existence of a class of well-fed non-loud-singing canaries. Then the non-loud-singing canaries which are asserted to exist must be both unhappy and ill-fed, and by eliminating the middle term we can reach the syllogistic conclusion that some ill-fed canaries are unhappy.

This example will show that Carroll did not accept the doctrine that has done so much to simplify traditional formal logic—the interpretation of a universal proposition as not involving the existence of its subject-term. " Canaries, that do not sing loud, are unhappy " involves, according to Carroll, the existence of some non-loud-singing canaries : without this assumption in this case no syllogistic conclusion can be drawn. Carroll defends his interpretation by remarking that, if an *A* proposition contains an *I* proposition, the former must be existential if the latter is. He only just notices (in a note at the end) that if *A* is existential, *O* cannot be if it is to contradict *A* ; so that the contradictory of all ill-fed canaries are happy would be either that there were ill-fed unhappy canaries or that there were no ill-fed canaries at all. And this would be grossly inconvenient. Carroll's lack of subtlety makes his system appear more satisfactory superficially than it is fundamentally.

The second volume of the *Symbolic Logic*, which was to deal among other things with hypothetical propositions, was never published. But he printed in *Mind* for 1894 a celebrated paradox about hypotheticals which led to a short and sharp controversy. The paradox is as follows : There are three barbers in a shop, Allen, Brown and Carr, whose movements are determined thus : (1) If Allen goes out, Brown has to go out too (because Allen is infirm) ; (2) all three cannot be out together (for business reasons). Now the second condition may be put in the form : If Carr goes out, then if Allen goes out, Brown stays in. But if Allen goes out, Brown goes out, by condition (1) ; therefore it may be argued that it is false that, if Allen goes out, Brown stays in ; therefore Carr never goes out at all. Whereas it is obvious that Carr can go out whenever Allen stays in. The solution of the paradox is to see that, whatever interpretation is given to the hypothetical proposition " If p then q," the proposition " If p then not $-q$ " is not incompatible with it. Both the propositions, " If Allen goes out, Brown goes out ", and " If Allen goes out, Brown stays in ", can be true at once—when

Allen is in. This article of Carroll's shows that he was thinking about the nature of hypothetical propositions, consideration of which has largely influenced recent developments in logic; and a connected subject is raised by Carroll's note in *Mind* of 1895 entitled "What the Tortoise said to Achilles". In this dialogue Achilles attempts to convince the Tortoise of the truth of the conclusion of a syllogism. The Tortoise is prepared to accept the two premisses, but will not accept the conclusion merely on the strength of them. He insists that "If the two premisses, then the conclusion" shall be inserted as a third premiss in the argument. And then, though he will accept this as true, as before he will not accept the conclusion on the strength of the three premisses alone. So a fourth premiss, "If the three premisses, then the conclusion", has to be added. And so on. Achilles, in his endeavour to persuade the Tortoise of the truth of the conclusion, has to add more and more premisses, and at every stage the Tortoise is unconvinced and demands another hypothetical proposition. This paradox shows very clearly the necessity in inference for an act of violence to cut the chain of hypothetical propositions—the need for what W. E. Johnson calls a "principle" as opposed to a "premiss" of inference, and for Whitehead and Russell's first primitive proposition —Anything implied by a true elementary proposition is true. Propositions asserting "implication" alone are not enough for inference. In both these papers in *Mind* Lewis Carroll was ploughing deeper than he knew. His mind was permeated by an admirable logic which he was unable to bring to full consciousness and explicit criticism. It is this that makes his "Symbolic Logic" so superficial (a fairly good dodge) and his casual puzzles so profound.

And it is this unconscious logic which is, I feel, the main reason for the supreme excellence of those unique works of genius—the two Alice books—and of what excellence there is in the two *Sylvie and Bruno's* and in the poems. Venn, a far better conscious logician than Carroll, in christening the barbers' shop paradox "Alice's problem", remarked that "the proposer is, to the general reader, better known in a very different branch of literature". This was superficial of Venn. The Alice books are, of course, works of imagination and as such cannot be compared with scientific treatises. But the thought behind the imagination is logical and not merely fanciful. I think that Carroll is considered a fanciful writer largely because of Tenniel's illustrations. I naturally think of the Jabberwock by means of a visual memory-image of Tenniel's picture. Without this the poem becomes a study in the creation of neologisms with no fanciful content. Nearly all Carroll's jokes are jokes either in pure or in applied logic. And this is one of the reasons why the books make such an appeal to children. When a child has learnt that a form of words is used in one particular context, he is surprised to find that it cannot be used in some other context; and the attractiveness of Carroll lies in the fact that his use of a phrase in an apparently correct but really nonsensical way appears as plausible to him as to the child. And so

there is no touch of that superiority which befouls so many books written by adults for children.

Even the horrible puns * to which Carroll was addicted are essentially logical. For a pun consists in using a word which has two or more senses in one sense when the other is expected, and its application in the syllogism has received considerable (though undeserved) consideration by logicians under the title of the Fallacy of the Ambiguous Middle. And the consideration of the more subtle misuses of words is an important part of the province of logic. Many different sorts of these are exemplified in Carroll's nonsense books. Indeed, philosophers have been able to find quotations from them to head most of their chapters. When P. E. B. Jourdain published in 1918 a witty commentary on various points of mathematical logic under the title of "The Philosophy of Mr. B*rtr*nd R*ss*ll", he added an Appendix of Leading Passages from the Alice books. Indeed in Cambridge it is now *de rigueur* for economists as well as logicians to pretend to derive their inspiration from Lewis Carroll.

I will call attention here to some of Carroll's logical points that I have not seen noted elsewhere and which raise some particularly fundamental problems. At the banquet towards the end of *Sylvie and Bruno Concluded* the Other Professor arrives reading a book and consequently trips up and falls on his face in the middle of the table :

"*What* a pity!" cried the kind-hearted Professor, as he helped him up.
"It wouldn't be *me*, if I didn't trip", said the Other Professor.

This argument of the Other Professor will be found in Joseph's *Introduction to Logic*, p. 408, in Bradley's *Appearance and Reality*, p. 364, in Whitehead's *Science and the Modern World*, p. 174, and in many other celebrated authorities. It is a subtle and dangerous fallacy, which tends to the monistic doctrine that everything depends logically upon everything else ; and it was only satisfactorily cleared up by G. E. Moore in 1919.

Another point which is relevant to recent logical work is the Professor's solution of Sylvie and Bruno's argument as to the length of the day : "My dears, the day is the same length as anything that is the same length as *it*". Jourdain took this merely as a statement of the Law of Identity : I think that the Professor had at the back of his mind the analysis by means of classes that has enabled modern logicians to dispense with absolute moments and intervals of time.†

* He was aware of their horror. "All Uncles make bad puns" is one of his syllogistic premisses. See also the reception of the King of Hearts' pun in the Trial at the end of *Alice in Wonderland*.

† A digression on to two points of philosophical psychology. The analysis of the taste of Alice's reducing bottle into "a sort of mixed flavour of cherry-tart, custard, pine-apple, roast turkey, toffee and hot buttered toast" anticipated Zwaardemaker's analysis of taste and Henning's of smell (I doubt whether Alice could discriminate between these) ; and Bruno's experiences with the Professor's black light were very similar to James Ward's. And a point in theoretical physics : entropy, which is alleged always to be greater yesterday and to-morrow than it is to-day, shares this property with the consumption of the White Queen's jam.

The Bellman's map (in *The Hunting of the Snark*) which had " no conventional signs " and was " a perfect and absolute blank " shows the conventions required for the use of coordinates : the Bellman may dispute with W. K. Clifford the honour of being the first General Relativist (both expositions were published in 1876). But the map described on p. 169 of *Sylvie and Bruno Concluded* has more logical interest :

" We actually made a map of the country, on the scale of *a mile to the mile* ! "

" Have you used it much ? " I enquired.

" It has never been spread out, yet ; the farmers objected : they said it would cover the whole country, and shut out the sunlight ! So we now use the country itself, as its own map, and I assure you it does nearly as well."

Can a symbol be used to symbolize itself ? Or is this just nonsense ? This is the profound question raised here. It will be found that nearly every paradox in the books, and every one that is of logical importance, is expressed in a sentence which, like the Hatter's remark, " seemed to have no sort of meaning in it, and yet it was certainly English ". Logic has two closely related tasks—the analysis and criticism of inference, and the analysis of the propositions or judgments which are used in inference. This latter task might be called the analysis of sense, or " contrariwise ", the analysis of nonsense. Here the Red Queen shows herself the prototype of the logician. To Alice's " A hill *can't* be a valley, you know, that would be nonsense ", she replies : " You may call it ' nonsense ' if you like, but *I've* heard nonsense, compared with which that would be as sensible as a dictionary ! " This is exactly the logician's trouble : some nonsense is as sensible as a dictionary, and it is the logician's business to show how nevertheless it is really nonsense. One of the difficulties is about " logically impossible " propositions (*e.g.* $2+2=5$) ; if these are sense, why is it that whenever we try to consider them, we find we are considering something else ? The White Queen, who alleged that she could believe " impossible things ", was surely correct if the things were sensible propositions.

Humpty Dumpty is the masterpiece of Lewis Carroll as unconscious logician. Dumpty is perhaps a little too sceptical about arithmetic, when he demands to see $365+1=366$ put into writing ; though he is spiritually in agreement with those mathematical logicians who insist that the meaning of a mathematical proposition is its proof. But it is his uncompromising attitude to words that makes him such a logical prophet :

" When *I* use a word, it means just what I choose it to mean—neither more nor less."

" The question is," said Alice, " whether you *can* make words mean different things."

" The question is," said Humpty Dumpty, " which is to be master —that's all."

Humpty Dumpty must take an honourable place among those who have attempted to free us from the bondage of symbols which are our own creation.

R. B. BRAITHWAITE.

ON THE OPTICAL WRITINGS OF SIR WILLIAM ROWAN HAMILTON.

By Professor G. C. Steward.

The study of Geometrical Optics has not attracted much interest or attention for some years now, at all events until recently, save only among a quite small company of specialists, and, as but one particular result of this, the divorce between optical theory and optical practice has been lamentable; as, for example, in that problem of so great practical importance, the design of the symmetrical optical system. For the one part, the subject has been regarded as isolated and of little theoretical interest, and scarcely susceptible of wide and profound generalisations; then, too, and this more especially from the point of view of university teaching, geometrical optics has seemed to stand apart from the path to a general mathematical education and training, and, in consequence, in several universities the study of it has been almost abandoned, save only in its most elementary aspects. It is, then, of no small interest to observe once more how profound and how wide were the generalisations to which geometrical optics lent itself in the hands of Sir William Rowan Hamilton, and to remember that his study in this field was the precursor, and indeed the parent, of the widely known and fundamental researches which he conducted in the realm of theoretical dynamics. Hamilton's method in dynamics was a direct development of his investigations in purely geometrical optics; indeed, in the dynamical developments he made use only of a particular case of his optical theory, so that the optical writings were themselves more general in nature and in scope than the succeeding dynamical work, although the dynamical theories and methods are far more familiar and more widely known than the optical studies which, in the mind of Hamilton, gave them birth.

We welcome then the knowledge that, at long last, some sixty-five years after his death, the collected works of Sir William Rowan Hamilton are to be published, and, in particular, we welcome this,* the first, volume of the collection, devoted to his researches in the realm of geometrical optics. The whole edition appears under the able and joint editorship of Professor A. W. Conway and Professor J. L. Synge, and we would pay a tribute here to the manner in which these editors have performed their task, and express also our hope that the remaining three volumes, which, they tell us, are forecast, will appear as soon as may be. For this first volume is prefaced by a concise, able and exceedingly valuable presentation of Hamilton's optical work, in general outline, showing carefully and clearly the genesis and development of his ideas in this realm. And, too, there is an editor's Appendix containing some twenty-seven notes of general explanation and comment; one indicating the precise connection between Hamilton's methods in optics and his methods in

* The Mathematical Papers of Sir William Rowan Hamilton: Volume I, *Geometrical Optics*. Edited for the Royal Irish Academy by A. W. Conway and J. L. Synge. Pp. xxviii, 534. 50s. 1931. (Cambridge University Press.)

theoretical dynamics, another dealing with the relation of each of these to the modern wave-mechanics, another on the computation of the aberration-coefficients for an optical instrument of revolution; and so on. Altogether the editors have undertaken a great and arduous task, and they have accomplished it worthily. And the resulting volume is a handsome one, surpassing, as we venture to think, anything which even the Cambridge University Press has hitherto produced.

The text of the volume, some four hundred and sixty large pages, is divided into three parts: Part I contains the celebrated essay on the *Theory of Systems of Rays* and the three *Supplements* to this, together with *Part Second* of the essay, not previously published; some three hundred pages of mathematical investigation which, as the editors comment, are not easy to read. In Part II are some of Hamilton's minor papers, variously published, while in Part III is given a selection taken from Hamilton's manuscripts and note-books which, before this, have never been published at all. These contain much matter of very considerable interest, matter which modifies to no small extent the commonly accepted view of Hamilton and his interests. And there are at present, we are told, in addition to unbound manuscripts, more than two hundred of Hamilton's note books in the Library of Trinity College, Dublin.

The fundamental and everywhere dominating thought throughout the whole of Hamilton's optical writings is, of course, that of the *characteristic-function* which he himself introduced, and it is of no small interest to observe that this function had an analytical origin and genesis. He considers a system of rays of light diverging from a single luminous point and reflected at the surface of a single curved mirror, and he proves, on the basis of the usual elementary laws of reflection of light, that, if x, y, z be the co-ordinates of any point upon, and α, β, γ the directions cosines of, a ray belonging to the reflected system, then the expression $\alpha\,dx+\beta\,dy+\gamma\,dz$ is a perfect differential, which, accordingly, he writes

$$\alpha\,dx+\beta\,dy+\gamma\,dz=dV,$$

so that $\alpha=\partial V/\partial x$, $\beta=\partial V/\partial y$ and $\gamma=\partial V/\partial z$. Here, then, is the first appearance of the characteristic-function V, a function of *position* x, y, z, its space-differential coefficients giving at once the *direction* of the ray passing through any assigned point. It is to be noticed that V is a function of three independent variables x, y, z—the luminous origin of the system of rays being regarded as given—and further that, because of the identical relation subsisting between the three direction cosines of any straight line, V satisfies a partial differential equation of the first order and second degree, namely,

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=1. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

V here is, of course, the distance to a general point x, y, z, from the given luminous source, measured along the ray which joins these two points. Hamilton proceeds then to extend this result to cover

the case in which the system of rays is any number of times reflected at successive mirrors, and, still further, any number of times refracted, in passing from one optical medium to another ; the origin of the system being still a given luminous point. It is evident, then, that the function V sums up in itself the behaviour of a *given* system of rays passing through a *given* optical combination. At this point we may pass in thought to the limiting case in which the optical media vary continuously and the ray becomes curved instead of polygonal, and we are dealing with a heterogeneous, but isotropic, optical medium. V then appears as an action integral,

$$\int_{P'}^{P} v\, ds,$$

taken along the ray from the luminous source P' to the general point $P(x, y, z)$; here v is a function of position, depending at each point upon the optical properties of the medium at that point. On an emission theory v is the velocity of the light corpuscle, and on an undulatory theory it is the reciprocal of the velocity of the wave, along the element of length ds, which, because the medium is isotropic, is normal to the wave-surfaces intersecting it.

The Third (and final) Supplement may be regarded as a complete and general statement of Hamilton's optical theory, without reference to the earlier papers. Here the restriction to an initial luminous point-source is removed and *any* incident system of optical rays is considered. The characteristic-function V appears at once as an action integral taken along the ray joining any two points, the one $P'(x', y', z')$ belonging to the initial region, and the other $P(x, y, z)$ to the final region ; thus

$$V = \int_{P'}^{P} v\, ds,$$

where v is a " medium function ", characterising optically the medium in which the element of path ds is measured. Moreover the media, all or any of them, may be anisotropic and also heterogeneous ; that is to say, v is to be regarded as a function, not only of the *position* of the element ds, but also of the *direction* of this element. The *form* then of the characteristic-function depends only upon the properties and arrangement of the optical media, and not upon the particular system of optical rays which traverse these media, and Hamilton proceeds to show that the behaviour of all possible systems of optical rays which may traverse these media may be deduced from the knowledge of the single function V.

By the usual rules of the Calculus of Variations, combined either with the principle of *stationary action* or with the principle of *stationary time*, the integral defining V may be varied, compared, that is to say, with its value taken along the *actual* optical path joining neighbouring terminal points, and we have (for light of one ' colour ' only)

$$\delta V = \frac{\partial v}{\partial \alpha}\delta x + \frac{\partial v}{\partial \beta}\delta y + \frac{\partial v}{\partial \gamma}\delta z - \frac{\partial v'}{\partial \alpha'}\delta x' - \frac{\partial v'}{\partial \beta'}\delta y' - \frac{\partial v'}{\partial \gamma'}\delta z'. \quad \ldots\ldots(2)$$

The function V, then, is to be regarded as a function of *six* vari-

ables, namely, x, y, z, x', y', z', which determine respectively the points P and P' of the final and initial system of rays ; and of *one* other variable, χ, specifying the ' colour ' of the light considered.

It has been assumed that $\delta V = 0$ for *fixed terminal points*, that is, for a slightly varied and *geometrically* possible path between P' and P ; that is to say, upon an emission theory, the principle of *stationary action*, or, upon an undulatory theory, the principle of *stationary time*. Hamilton is at pains, time and again, to point out that he is making no decision between the corpuscular and the undulatory theories of the nature of light (whatever, as he says, light may be)—a matter of controversy in his day ; his methods, theories and results hold equally in either case. For upon the former hypothesis $\delta \int v\, ds = 0$, for a (constrained, or *geometrically* possible) path variation between fixed terminal points, the principle of stationary action, where v is now at each point the velocity of the light corpuscle, and upon the latter hypothesis $\delta \int v\, ds = 0$, again for a (constrained) path variation between fixed terminal points, the principle of stationary time, where v is now the *reciprocal* of the *ray* velocity ; either principle therefore leads to Hamilton's basic result expressed by equation (2), which is " the Equation of the Characteristic Function " . . . " the variation of the action, or the time, expended by light of any one colour, in going from one variable point to another " . . . and " involves very various and extensive consequences, and appears to me to include the whole of mathematical optics ".

The whole of Hamilton's optical theory follows from equation (2) ; and we may write the equation as follows :

$$\delta V = \sigma\, \delta x + \tau\, \delta y + v\, \delta z - \sigma'\, \delta x' - \tau'\, \delta y' - v'\, \delta z', \quad \ldots\ldots\ldots\ldots (3)$$

where $\sigma = \partial v / \partial \alpha = \partial V / \partial x, \ldots$, and $-\sigma' = \partial v / \partial \alpha' = \partial V / \partial x', \ldots$. If, as a special case, the end media be isotropic, but not of necessity homogeneous, and if μ' and μ be their ' medium functions ' (refractive indices), then

$$\sigma = \mu\alpha, \ \tau = \mu\beta, \ v = \mu\gamma \quad \text{and} \quad -\sigma' = \mu'\alpha', \ -\tau' = \mu'\beta, \ -v' = \mu'\gamma',$$

so that we can deduce at once, from the characteristic-function V, the *direction* of the ray passing through any assigned point. Again, from the three equations giving σ, τ and v, combined with the identical relation always connecting the direction cosines of any straight line, we may eliminate these direction cosines and so obtain a relation between σ, τ, v, x, y and z, that is, a differential equation of the first order in V, which may be written

$$\Omega\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}, x, y, z\right) = 0.$$

Equation (1) is, of course, a special case of this. And there is a similar differential equation satisfied by the characteristic-function in the initial medium, namely,

$$\Omega'\left(\frac{\partial V}{\partial x'}, \frac{\partial V}{\partial y'}, \frac{\partial V}{\partial z'}, x', y', z'\right) = 0.$$

In general, in media which are anisotropic, the rays are not normal to the surfaces $V =$ constant, and, upon an undulatory theory, v is

the *reciprocal* of the ray-velocity and σ, τ and υ are the *components of normal slowness*, that is to say that $(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$ is the normal velocity of the element of the wave-surface given by $V=$ constant.

In this Third Supplement Hamilton defines in a general manner, and in the earlier papers in a more special way, two important associated functions, which appear as modifications of the characteristic-function V. Thus, if, as a particular case, we consider isotropic end media, we have $\sigma=\mu\alpha, \ldots, -\sigma'=\mu'\alpha', \ldots$, as above, and we may write

$$T+V=\sigma x+\tau y+\upsilon z-\sigma' x'-\tau' y'-\upsilon' z',$$

so defining the function T. By variation of this, and in virtue of the preceding relations between $\sigma, \ldots$, $\sigma', \ldots$ and the differential coefficients of V, we have

$$\delta T=x\,\delta\sigma+y\,\delta\tau+z\,\delta\upsilon-x'\,\delta\sigma'-y'\,\delta\tau'-z'\,\delta\upsilon',$$

so that T may be regarded as a function of the quantites $\sigma, \ldots$, $\sigma', \ldots$, its differential coefficients with respect to these quantities giving at once the co-ordinates $x, \ldots, x', \ldots$; thus T is a function of the *directions* of the ray passing through the final and initial points chosen. For isotropic and homogeneous end media we have

$$\mu x=\frac{\partial T}{\partial a}, \ldots, \quad -\mu' x'=\frac{\partial T}{\partial a'}, \ldots,$$

where now the rays, at each end, are straight lines. It is of interest to observe here that Hamilton's function T is identical, save only as to sign, with the Eikonal-function, E, introduced by Bruns some sixty years later. In fact, Hamilton himself makes use elsewhere of precisely this function E, under the symbol S. Bruns, we understand, was familiar only with Hamilton's subsequent dynamical researches, which proceeded from the optical writings, and so obtained his Eikonal-function.

Similarly a function W is derived from the equation

$$W+V=\sigma x+\tau y+\upsilon z,$$

so that W is a function of the six quantities $\sigma, \tau, \upsilon, x', y', z'$; that is, for homogeneous and isotropic end media, of the *initial co-ordinates* and *final direction cosines*. Hamilton gives many transformations of these functions, tracing carefully the relations between them, and between their differential coefficients of various orders with respect to the variables upon which they severally depend; and examines the sudden changes in the T and W functions consequent upon reflection, or upon refraction whether ordinary or extraordinary. Moreover he develops the methods according to which he applied them afterwards, without however publishing his results, to an investigation of the properties of the symmetrical optical system.

The preceding very brief sketch deals only with the basic functions which Hamilton introduced into the study of geometrical optics; but many purely geometrical results, too, occur in these writings. In a homogeneous and isotropic medium the rays of light form a doubly infinite system of straight lines, normals to the family of

surfaces given by $V = \text{constant}$, that is to say, these lines form a *rectilinear congruence*. Accordingly Hamilton considers many properties of such a congruence—the two foci of each ray and the coalescing of these, for a finite number of rays, into one principal focus, their loci the caustic surfaces, and various properties of developable and of undevelopable pencils of rays; moreover, he examines the 'density' of the distribution of the rays. And further, considering any assigned ray, and a plane normal to it, he investigates the intersections of this plane by the neighbouring rays, first for a general normal plane, then for one passing through one of the foci of the ray, and finally for a plane passing through a principal focus (when the ray possesses such). In this manner he investigates thoroughly the (first order) 'aberrations', and this, it is to be noticed, for an optical system which is not in any way symmetrical; dealing from this very general point of view with what is effectively first order 'coma'. Then, passing on, the curved rays, forming a doubly infinite system in an isotropic but heterogeneous medium, are examined, as to whether they intersect, and, if they do, the loci of their intersections and the various properties of these 'caustic surfaces'; here again the rays, although curved, are all normals to a family of surfaces, given by $V = \text{constant}$. And, finally, in the great Third Supplement, Hamilton investigates the properties of rays in anisotropic and heterogeneous media, so generalising completely his preceding work; here, of course, the rays are no longer normals to the family of surfaces $V = \text{constant}$ (the wave-surfaces on an undulatory theory, or, upon an emission theory, the action-surfaces); there is both a *ray-velocity* and a (different) normal velocity of the surface element. Moreover, there are both the curved rays, a doubly-infinite system, and also the *ray-lines*, tangents, that is to say, to the rays and forming a triply infinite system of straight lines, or a *complex* of lines. And Hamilton investigates, in a very general manner, the general relations of infinitely near rays; and especially (and as usual) how the extreme directions change, for any infinitely small changes of the extreme points and of the 'colour'. Here he deals with the 'points of vergency' and their determination, their loci the caustic surfaces and the arrangement and aberrations of the rays near these foci. There are, then, many results of purely geometrical interest, some of which are now incorporated in the text-books of geometry and dealt with here, for the most part, upon the basis of the characteristic-functions which Hamilton had introduced earlier in these memoirs; and Kummer, often regarded as the founder of the general theory of the rectilinear congruence, is careful to give full credit to the work of Hamilton. Finally, by applying his functions and methods to the undulatory theory of Fresnel, he shows theoretically that in certain circumstances (in crystalline media) a single ray should give rise to a hollow cone of rays, a result then unknown but afterwards verified experimentally, at Hamilton's request, by Professor Phillips. Here, then, is the prediction and discovery of *conical refraction*, sometimes looked upon as the culminating point of his theory, but, as the editors

comment, better regarded as but one particular application of a powerful mathematical method—that of the characteristic-function.

It may not be out of place, perhaps, to repeat here that the fundamental contribution which Hamilton made to optical investigations, and through these to dynamics and so to the whole range of physical science, was the *Law of Varying Action* expressed in his basic equation (2) preceding. He took the principle of *stationary action*, or of *stationary time*, each of which may be expressed symbolically by $\delta \int v\, ds = 0$, where v is, in the one case, a particle velocity, or, in the other case, the reciprocal of a ray velocity along the element of path ds; and the sign of variation δ indicates that a comparison is made between the value of the preceding integral taken along the *actual* optical (or dynamical) path between fixed terminal points P' and P, with the value of the same integral taken along a curve, everywhere near to the former path and joining the same terminal points—a curve, then, which is neither optically nor dynamically possible as a free path, but is a ' constrained ' path; and this, incidentally, has been widely associated with the idea of ' economy ' in nature. The principle is connected with the names of Fermat and Maupertuis, and was developed by Euler, by Lagrange and by Bernoulli; Hamilton, assuming these results, compares the value of the integral $V(=\int v\, ds)$ taken along the actual optical (or, as later, the dynamical) path between the terminal points P' and P with the value of the same integral taken along the actual optical (or dynamical) path between neighbouring terminal points Q' and Q—so comparing the values of the integral for neighbouring *free* paths. And he shows that the corresponding variation depends (to the first order) only upon the (small) changes in the positions of the terminal points themselves and upon the optical properties of the media in the neighbourhood of these points; in this way reciprocal relations are obtained between two distant stages of the progress of the luminous disturbance. The motion of a dynamical system of n degrees of freedom is completely represented by the optics (upon an emission theory) of a medium of n dimensions, and so we are led to reciprocal relations between any two distant stages of the progress of the dynamical system under investigation.

Amongst the minor papers reprinted in Part II of the volume we notice in particular a paper *On a General Method of Expressing the Paths of Light, and of the Planets, by the Coefficients of a Characteristic Function*; here Hamilton gives a brief, but exceedingly interesting, review of optical theories from earliest times, and more particularly of the emission theories of Newton and the undulatory theory of Huygens, of their coming together in the mind of Maupertuis, and of the consequent emergence of the dynamical principle of *least action* (preferably *stationary action*). Hamilton felt the need of some such general method in mathematical optics, and he tells us how he found that method in this action principle, or rather in the *Law of Varying Action*, in which form the principle appears in his hands. He gives a number of very simple applications of each principle, to the straight line regarded as the shortest distance between two given points,

through various cases until the characteristic-function appears in all its generality, as applying to all optical media ; and finally he gives a brief adumbration of his subsequent use of this function in dynamical investigations.

It is abundantly clear, we think, from the preceding general, but very brief, outline of the memoir on the *Theory of Systems of Rays* and of the *Supplements* to this, that the functions which Hamilton introduced there (the characteristic-function V and its modifications W and T) afford of themselves sufficient aid for the investigation of all the purely geometrical phenomena associated with all optical systems whatsoever, whether all, or any, of the media involved be isotropic or anisotropic, homogeneous or heterogeneous, and for light of any ' colour '—and all this whether an undulatory theory be assumed or a corpuscular theory ; in particular, they present a ready method of examining, theoretically, the performance of the symmetrical optical system, a case of great practical importance. Hamilton several times expresses his intention of applying himself to this problem, but nowhere in his published papers does any examination of it appear ; yet in the manuscript of his note-books, now published for the first time, he sets himself resolutely to examine the symmetrical system, both from the theoretical and from the numerical points of view. And to the consideration of these newly revealed results of his we proceed now. It is interesting to notice, however, that during the years succeeding the publication of Hamilton's memoirs but few attempts were made to use his methods in problems concerning geometrical optics, and these attempts were along the line of general and qualitative application only. We may mention here the work of Maxwell, of Kirchhoff, of Rayleigh, of Larmor, of Bruns above mentioned, of Schwarzschild and of Bromwich, and also a small book by Pelletan ; but in no case were the investigations pursued as far in the practical direction as, in his unpublished writings, Hamilton himself had proceeded. Indeed the main charge brought by ' practical ' men against the use of Hamilton's functions was that, although theoretically complete and general, indeed because of their great generality, they were of no practical value on account of the complexity of their actual forms when applied to any particular optical system. And, in fact, the view has been widely held that Hamilton's sole interest lay in establishing general methods of attacking problems, and not in the investigation of special questions ; a view which may quite legitimately be inferred, perhaps, from a perusal of the general memoirs which alone Hamilton published, but which is shown to be altogether erroneous by the note-book extracts now published for the first time. Hamilton, indeed, had a mind of exceedingly great generality, but, also, he could apply his most general theories to particular cases of importance, indeed to some seeming almost trivial, a triviality, however, which served to bring into naked evidence the general methods in all their power.

In order to estimate the work of Hamilton in this realm, we may pause here, perhaps, to indicate, in briefest outline, a general method of procedure based upon his functions and developed during recent

years.* Thinking, then, of the symmetrical optical system, we may concentrate our attention upon Hamilton's T-function, a function of *six* variables, namely, the direction cosines, α, β, γ, of a final ray and those, α', β', γ', of an initial ray, referred to convenient axes of co-ordinates of which, in each case, the axes of z and of z' coincide with the axis of symmetry of the optical system ; and we deal with light of one 'colour' only and assume that the optical indices of the end media are each equal to unity. Owing to the identical relations existing between the direction cosines of any straight line, these six variables may be reduced to *four*, and further to *three*, because of the axial symmetry of the system, and, as these variables, we may choose conveniently the three quantities $\alpha^2+\beta^2$, $\alpha\alpha'+\beta\beta'$, $\alpha'^2+\beta'^2$. It is to be noticed that the function T embodies in itself the behaviour of *any* and *every* system of optical rays which may pass through the instrument of revolution. Now, thinking of two conjugate planes, each normal to the axis of the optical instrument, we may ask: what must be the form of the function T so as to ensure perfect imagery for these two planes, to ensure, in other words, point to point correspondence between the two planes so that to each cone of rays passing through any point upon either plane shall correspond a cone of rays passing through the corresponding point upon the other plane ? And it appears that the necessary and sufficient condition for this is that T shall be a function, say $u(\zeta)$, of a certain variable ζ, itself a linear function of the three variables indicated above. But such perfect imagery may be shown to be impossible and therefore T must depend also, for all actual optical systems, upon two other variables, say ξ and η, which are linear functions of the initial variables, and accordingly we may write

$$T(\xi, \eta, \zeta) = u(\zeta) + \Phi(\xi, \eta, \zeta) ;$$

so T appears as a function of the three new variables ξ, η and ζ. Here, then, the function $\Phi(\xi, \eta, \zeta)$ sums up in itself all the *aberrations* of the symmetrical system—the wandering of the various rays from their ideal foci—and depends upon these aberrations alone ; so that $\Phi(\xi, \eta, \zeta)$ may be named, appropriately, the *aberration-function*. The *form* of the function $u(\zeta)$ may be determined so as to satisfy other conditions. If we consider rays diverging from a luminous point upon one of the conjugate normal planes mentioned above, and if any such ray, after passage through the instrument, intersect the corresponding normal plane in a point x, y, referred to the ideal focus as origin, then it may be shown that

$$x = m\frac{\partial\Phi}{\partial\alpha} + \frac{\partial\Phi}{\partial\alpha'}, \quad \text{and} \quad y = m\frac{\partial\Phi}{\partial\beta} + \frac{\partial\Phi}{\partial\beta'}, \qquad \ldots\ldots\ldots\ldots(4)$$

where m is a constant, the paraxial magnification associated with the particular conjugate planes chosen ; and the differentiations

* An outline sketch of the properties of the symmetrical optical system, from the purely geometrical and also from the physical (diffraction) points of view, will be found in *The Symmetrical Optical System*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 25, by the present writer.

with respect to α, α', β and β' may be replaced by differentiations with respect to the new variables ξ, η and ζ on account of the linear relations above indicated. These relations (4) give then completely the aberrations for the conjugate planes chosen, and, by a slight extension, for any neighbouring plane. Further, Φ may be expanded, in general, in powers of ξ, η and ζ, and, accordingly, we write

$$\Phi = \Phi_2 + \Phi_3 + \ldots + \Phi_n + \ldots ,$$

where Φ_n is homogeneous and of degree n in ξ, η and ζ (themselves usually small quantities), and Φ_0 and Φ_1 have been omitted since, for the cases considered, the aberrations are of order two, or of higher orders, of small quantities. Hence then the suggestion of dividing these aberrations into groups of various orders of small quantities. Φ_2 will have *six* terms, one of them, that in ζ^2, being annihilated by the operators of (4), and we have then five ' first ' order aberrations, the so-called ' five aberrations of Von Seidel ' ; namely, *spherical aberration*, *coma*, *astigmatism*, *curvature of the field*, and *distortion*. It is not without interest to notice, perhaps, that four of these aberrations had been discovered and investigated in this country before the time of Von Seidel, by Herschel, Airy, and Coddington ; indeed, Airy was the discoverer of the formula for the flatness of field (to the first order) which commonly, following German writers, is associated with the name of Petzval, the so-called ' Petzval-sum '. But to Kirchhoff has usually been assigned the discovery of coma, which emerged from an investigation carried out by him by means of Hamilton's characteristic-function ; yet, as will appear shortly, Hamilton himself, in the hitherto unpublished papers, gives a very complete investigation of this aberration, and also its effects upon planes slightly ' out of focus '. As a matter of fact he had considered it in a very general way in a geometrical investigation buried in the *Theory of Systems of Rays*, dealing there with an optical system which was not in any way symmetrical. The ' five aberrations of Von Seidel ', then, had been all investigated in this country long before the time of Von Seidel.

Again the function Φ_n, where $n > 2$, is concerned with geometrical aberrations of a higher order, of order $n-1$; of which we have *nine* of the second order, *fourteen* of the third order, and more generally, $n(n+3)/2$ aberrations of order $n-1$. And the geometrical meanings of these have only recently been investigated with any fulness or in any general manner ; they give rise to an interesting series of aberration curves.* Other developments of interest can only be mentioned here ; the widely known *sine-condition* and its meaning, the very general and quite recently discovered *optical cosine law*,† which embodies, as special cases of it, not only the sine-condition and Herschel's condition, but also a number of other optical laws ; the quite different grouping of the

* Cf. *Trans. Camb. Phil. Soc.* xxiii, No. ix (1926).

† Due to T. Smith : cf. " The Optical Cosine Law ", *Trans. Opt. Soc.* xxiv (1922-3), No. 1 ; and, for another proof, *Proc. Camb. Phil. Soc.* xxiii, vi, p. 703, " On Herschel's Condition and the Optical Cosine Law ", by the present writer ; or, Camb. Math. and Phys. Tracts, No. 25, ch. iv, §§ 6 and 7.

aberrations, based upon certain invariant and semi-invariant properties which some of them possess—a matter of theoretical interest and also, we think, likely to be of practical importance (a field concerning which as yet but little has been written). Further, the asymmetrical optical system has been under investigation again recently.

The preceding outline has been concerned *qualitatively* with the symmetrical system; it is clear that, before these results can be applied practically, we must consider particular optical systems. And, in brief, the method is to express the coefficients of the various terms in ξ, η and ζ, appearing in the aberration function Φ—the *aberration coefficients*—as functions of the data of such particular systems, the curvatures and separations of the surfaces and the indices of the various optical media involved. These coefficients are obtained, in the first place, for a single surface of revolution (whether spherical or not) and then 'addition formulae' obtained, so that they may be 'added' from surface to surface and computed for any known optical combination. In this way all the aberration properties of any such combination are known—for all systems of rays traversing that combination.

When we turn to the investigations of Hamilton himself, now published for the first time, we are astonished to perceive how far he had proceeded in the application of his general methods and functions to particular problems concerning the symmetrical optical system, a project frequently mentioned in his general memoirs, upon which, however, he appears not to have published anything at all. In a manuscript on *The Aberrations of an Optical Instrument of Revolution* he applied his T-function to the qualitative examination of the general symmetrical system, in the first place as regards paraxial imagery, and then with reference to the first order geometrical aberrations. He examined carefully the (first order) aberrations known as *coma* and *astigmatism*; obtaining the usual coma-diagram and proceeding to investigate also the 'out of focus' coma effects, showing that the paraxial image plane is, for this aberration, a plane of symmetry; and he obtained, too, the limaçon * as the 'out of focus' coma curve. This, then, is a particular case, for a symmetrical system and for a central ray near to the axis of symmetry, of the general investigation,† dealing with *any* principal ray and an optical system entirely unsymmetrical, given in his main memoirs; so that here a (first order) coma circle replaces a (first order) coma

* Cf. *Trans. Camb. Phil. Soc.* xxiii, No. ix (1926), § 26, by the present writer.

† The usual treatment of (first order) coma is but a special case of Hamilton's investigation, given in § 60, sect. xii, on *Aberrations*, in Part I of the *Theory of Systems of Rays*; for here Hamilton considers the distribution of near rays around any ray having what he defines as a principal focus, *i.e.* a ray for which the usual two foci coincide; and the optical system is in no way symmetrical. He finds a family of ellipses enveloped by two straight lines passing through the principal focus; a somewhat similar phenomenon to that found by the present writer to hold for symmetrical systems and for aberrations of the second and higher orders, which he has named "elliptical coma" (*Trans. Camb. Phil. Soc.* xxiii, No. ix, § 8). And there are also more generalised types of coma.

ellipse. And this is the aberration the discovery of which has usually been attributed to Kirchhoff, at a much later date.

There are several manuscripts of a quantitative nature, and, in particular, one of some seventy-four pages under the title (assigned by the editors) *On the Improvement of the Double Achromatic Object Glass*; here Hamilton made a determined attack upon the actual (quantitative) calculation of the T-function, up to and including those terms upon which depend all the aberrations of the first order, for various symmetrical optical systems; for a single surface of revolution, for a lens, for a combination of two lenses, for any number of coaxal refracting surfaces of revolution (but not of necessity spherical) placed close together. The coefficients of the T-function are found here, of course, in terms of the data of the particular optical system under consideration, the curvatures and separations of the several surfaces and the optical indices of the various media involved. The whole is interspersed with many calculations and comments of interest and importance. There are given also the formulae upon which are based the numerical calculations by which he computed some telescopic object glasses, the specifications of which he sent to his friend Professor Phillips. Again there appear the first two of Von Seidel's conditions for the case of a thin system, the conditions for the absence of (first order) coma, the second, of course, being the first approximation to the well-known sine-condition; and, too, the condition for the flatness of the locus of the primary focus—and it will be remembered that the so-called 'Petzval' condition applies only when astigmatism is absent and the two foci coincide. Altogether there appears in this manuscript a great amount of work which can scarcely be indicated here, work of the greatest interest and importance to anyone dealing with the practical side of optical calculation, and indeed with the theoretical side too. As the editors comment, all this "finally exonerates Hamilton from the accusation of having (only) a purely theoretical interest in geometrical optics".

Hamilton's optical writings are concerned with systems of rays, as he says, with "the properties and relations of these linear paths of light . . . whether we adopt the Newtonian or the Huygenian, or any other physical theory . . . as an important separate study, and as constituting a separate science, called often *mathematical optics*". This assumes, of course, that the only requisite for a point image is that the emergent rays should themselves pass through a point; but owing to the physical nature of light (upon an undulatory theory) and to its finite, though relatively small wave-length, this assumption is untenable. For a part only of the incident wave will pass through the system, and for this reason diffraction phenomena will appear—so that in no case will there be a point image but, instead, a diffraction pattern, and this diffraction pattern is modified profoundly by the presence of the purely geometrical aberrations caused by the passage of the wave through the optical system. The investigation of these diffraction patterns is a problem which, of recent years, has attracted some attention. Now Hamilton

himself was, of course, sensible of all this, but he definitely omitted this realm: "... it is not here intended to include the *phenomena of interference*, yet it is easy to perceive, from the nature of the quantity which I have called the characteristic-function, and which in the hypothesis of undulations is the *time of propagation of light from one variable point to another*, that the study of this function must be useful in such investigations also". The problem has been examined in recent years, seeking the diffraction effects of the separate aberrations one by one, and based directly upon their purely geometrical properties. But surely Hamilton's characteristic-function offers by far the readiest and most easy method of procedure, summing up, as it does, in the form of it, all the geometrical aberrations and their properties, and giving at once the time of the propagation of the optical disturbance. And the present writer may be permitted, perhaps, to refer to an investigation * which he himself has carried out, dealing with the diffraction effects of the first order geometrical aberrations in the region near the axis of a symmetrical optical system; and much yet remains to be accomplished with regard to the region away from this axis and including also the (very marked experimental) effects of the higher order geometrical aberrations; and of course also with regard to non-symmetrical systems.

Associated even with the words of a writer such as Sir William Hamilton there is something which itself cannot be reduced to words, something which must be experienced by anyone reading his memoirs and writings; the touch everywhere of a master hand. Hamilton had a mind of exceeding great generality, a mind which could yet apply its most general conceptions to particular problems—some of which were themselves of a very general nature, others seeming almost trivial, while yet serving to bring more fully into evidence, and shorn of all analytical complexity, the power and vastness of his general ideas. And the reader cannot fail to be impressed by his style of writing and by the manner in which he pursues his researches to their completion and draws forth from his general results their many implications. A careful perusal of these hitherto unpublished manuscripts throws, too, an interesting light upon Hamilton's method of working; for one sees the unsparing toil by which, through many a particular case, the general methods were evolved, to be presented finally in the complete and compact form of the great memoirs. Truly the reader may agree with some words of the Eloge delivered in 1865 by the President of the Royal Irish Academy, "... the fame of Sir William Rowan Hamilton, great as it was during his lifetime, will become yet greater when the world has been furnished with materials enabling it more perfectly to estimate the variety and richness of his endowments and the value of the services which he has rendered to Science". G. C. S.

860. How much courage and self-confidence it needs, when one comes to look into it closely, to undertake to teach.—A. Tchekov, *Tales*, trans. C. Garnett, xii, p. 75.

* *Phil. Trans. Roy. Soc.* A, 225 (1925).

AN EXTENSION OF DETERMINANTS.

By A. Lees, B.Sc.

If the elements of a determinant of order n are given by

$$a_{rs}(r, s = 1, 2, \ldots n),$$

then the rule for determining the sign to be affixed to any term of the determinant is usually given in a form equivalent to the following: arrange the n elements of the term with one set of suffixes in ascending order (*i.e.* 1, 2, ... n), and count the number of inversions, equal to i, say, in the other set. Then the correct sign is $(-)^i$.

It is, however, easily seen that i is equal to the sum of the numbers of inversions in *each* set, the elements being arranged in any order whatever.

Now if there were three suffixes instead of two, we could not define i as the sum of the numbers of inversions in each set, for an interchange of two consecutive elements would change i by ± 3 or ± 1, so $(-1)^i$ would change sign. But this is because 3 is an odd integer. If we have any even number of suffixes, i can be defined as above, and $(-1)^i$ has a definite value, either $+1$ or -1.

We accordingly define the "determinant" of order r and of $2n$ dimensions, whose elements are given by $a_{\alpha\beta\ldots\kappa}(\alpha, \beta, \ldots \kappa = 1, 2, \ldots r$, there being $2n$ suffixes) as the sum of all products of the elements taken r at a time, such that each of the numbers $1, 2, \ldots r$ appears once (and only once) as a first suffix, once as a second suffix, etc., in each product, the sign to be attached to each product (or "term") being determined as above.

When $n = 1$, this reduces to an ordinary determinant. The number of elements is obviously r^{2n}. The number of terms in the expansion is equal to r^{2n-1} multiplied by the number of terms containing a given element whose first suffix is, say, 1. For each term contains one and only one such element, and there are r^{2n-1} such elements.

But the cofactor (with the usual meaning) of the element is a $2n$-dimensional element of order $r-1$.

Therefore, if $\emptyset(r, n)$ be the number of terms in a $2n$-dimensional determinant of order r,

$$\emptyset(r, n) = r^{2n-1}\emptyset(r-1, n).$$

Similarly,
$$\emptyset(r-1, n) = (r-1)^{2n-1}\emptyset(r-2, n),$$

.....

$$\emptyset(2, n) = 2^{2n-1}\emptyset(1, n).$$

Also clearly
$$\emptyset(1, n) = 1.$$

Therefore, multiplying up and cancelling common factors,

$$\emptyset(r, n) = (r!)^{2n-1},$$

giving the number of terms in the expansion. Corresponding to rows and columns in ordinary (two-dimensional) determinants, we have sets of elements, any one set consisting of the r^{2n-1} elements having one suffix common. Two sets determined by giving the same suffix two different values will be called "similar". (In ordinary

determinants, two similar sets will both be rows or both be columns.)

The following theorems are obvious generalizations of the elementary theorems for two-dimensional determinants, and the proofs are very similar. By Δ we mean the value of our generalized determinant.

(α) If corresponding elements in two similar sets be interchanged, Δ is changed in sign.

(β) If corresponding elements in two similar sets are in a constant ratio, $\Delta = 0$.

(γ) If constant multiples of the elements of any set be added to the corresponding elements of a similar set, Δ is unchanged.

(δ) If the elements of any set are multiplied by λ, Δ is multiplied by λ.

(ϵ) If two dissimilar types of set be interchanged (*i.e.* two suffixes interchanged), Δ is unchanged.

(This last theorem corresponds to "interchanging rows and columns" in the two-dimensional case.)

There is a multiplication theorem for generalized determinants, corresponding to that for ordinary determinants, but rather more complex. Accounts of it will be found in the papers mentioned below.

Ordinary determinants are important both as a practical method of solving systems of linear equations and for their general theoretical significance, particularly in connection with the theory of covariants and invariants. Unfortunately, the former aspect does not admit of any profitable extension to $2n$ dimensions, but the latter is easily generalized into a systematic method of determining covariants. We give two examples of this. Firstly, we generalize the Jacobian and Hessian, which are well known types of covariant. The Jacobian and Hessian are represented by the determinants

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_r} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_r}{\partial x_1} & \frac{\partial u_r}{\partial x_2} & \cdots & \frac{\partial u_r}{\partial x_r} \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_r} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \frac{\partial^2 u}{\partial x_2^2} & \cdots & \frac{\partial^2 u}{\partial x_2 \partial x_r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 u}{\partial x_r \partial x_1} & \frac{\partial^2 u}{\partial x_r \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_r^2} \end{vmatrix}.$$

These are both particular cases of the following general type of covariant. Let there be r^m functions $u_{\alpha\beta \dots \kappa}(\alpha, \beta, \dots \kappa = 1, 2, \dots r$, there being m suffixes) of r variables $x_1, x_2, \dots x_r$. Then our covariant is a determinant of order r and $2n$ dimensions whose typical element is $b_{\alpha\beta \dots \kappa\lambda \dots \zeta}$ (there being $2n$-m suffixes $\lambda \dots \zeta$) where

$$b_{\alpha\beta \dots \kappa\lambda \dots \zeta} = \frac{\partial^{2n-m}}{\partial x_\lambda \dots \partial x_\zeta} u_{\alpha\beta \dots \kappa}.$$

If $m = n = 1$, we have the Jacobian of a number of functions. If $m = 0$ and $n = 1$, we have the Hessian of a single function.

As a second example, consider the invariant J of the quartic

$a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4$. It can be expressed as

$$J=\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

Except for a constant factor, this is the same as

$$\begin{vmatrix} \frac{\partial^4 u}{\partial x^4} & \frac{\partial^4 u}{\partial x^3\partial y} & \frac{\partial^4 u}{\partial x^2\partial y^2} \\ \frac{\partial^4 u}{\partial x^3\partial y} & \frac{\partial^4 u}{\partial x^2\partial y^2} & \frac{\partial^4 u}{\partial x\partial y^3} \\ \frac{\partial^4 u}{\partial x^2\partial y^2} & \frac{\partial^4 u}{\partial x\partial y^3} & \frac{\partial^4 u}{\partial y^4} \end{vmatrix}, \text{ where } u=a_0x^4+\dots .$$

We generalize this into a determinant of order r and of $2n$ dimensions, whose typical element $a_{\alpha\beta\dots\kappa}$ is given by

$$a_{\alpha\beta\dots\kappa}=\frac{\partial^{2n(r-1)}u}{\partial x^{2nr-\Sigma\alpha}\partial y^{\Sigma\alpha-2n}},$$ u being any function of x and y.

If $r=3$ and $n=1$, we get the above determinant. If $r=2$ and $n=2$, and we take u to be the quadric expression, we easily arrive at the invariant I of the quadric, *i.e.* $a_0a_4-4a_1a_3+3a_2^2$.

These examples are but a small indication of the importance of multi-dimensional determinants. In addition to their bearing on the theory of covariants, they are closely connected with the calculus of tensors and the theory of matrices, both of which subjects are of fundamental importance at the present time.

At the time of writing this paper I believed the results to be new, but I have since learned that this is not so. However, the fact that the subject is so little known may justify this article. Further accounts of the theory of n-dimensional determinants will be found in the following papers (I have to thank Professor H. W. Turnbull, M.A., F.R.S., for these references): L. H. Rice, "*P*-way Determinants, with an Application to Transvectants" (*American Journal of Mathematics*, vol. 40, p. 242): R. Vaidyanathaswamy, "On Mixed Determinants" (*Proc. Roy. Soc. Edin.*, 44, p. 168). Both these writers consider determinants of odd as well as even dimensions. This is done by introducing a certain lack of symmetry into the affixing of the signs of the terms by neglecting an odd number of suffixes, so that an even number remain.

A. Lees.

University College, Nottingham, *May* 1931.

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861. Question: Find the value of $\sqrt[3]{(4{\cdot}156\times 0{\cdot}00612\div 0\,0891)}$. A boy worked correctly as far as finding the logarithm of the quantity under the radical to be $\bar{1}{\cdot}4555$. He then (i) forgot the root, (ii) looked up 4555 in the log table instead of the antilog table, and obtained the answer 0·6585, which is right to four figures!

If t stands for the answer, this means that

$$\log\{10(3\log t+1)\}=t,$$

which appears to have only two solutions, $t=1$ and $t=0{\cdot}6585$. [Per Mr. C. W. Adams.]

MATHEMATICS IN THE SCHOOLS.

By A. N. Hickling.

The teaching of mathematics has been brought to so high a pitch of excellence by the eminent teachers of the past century that it may be thought there is little scope for improvement in their methods.

Since the radical revision of our geometry text-books towards the end of the nineteenth century, when modern methods first superseded the age-old system of Euclid, we find no fundamental change of doctrine until we come to consider the place of trigonometry and of the calculus in present-day curricula.

Before passing to a detailed consideration of the trend of contemporary thought on the scope of mathematics in the schools and as to how far its various branches should be comprehended within their curricula, a few words in regard to the teacher himself may not be out of place.

The mere mechanical appliances of the teacher, if I may so term books, however excellent in themselves, will play a small part in the instruction of the young pupil. Until matriculation standard has been reached, a good teacher will depend very little upon his text-books other than as a source of examples, except perhaps in regard to geometry. He will rely entirely upon his own exposition while adhering more or less closely to the course of instruction laid down in the text-book in use. At so early an age boys will pay scant attention to so uninteresting a book as a mathematical text necessarily forms. Thus at this early stage, when the boy will in all probability acquire a permanent taste or distaste for the subject, everything depends upon the master. If his teaching is lucid and forceful, his reasoning logical and vigorous, then the foundations of the boy's mathematical knowledge will be well and truly laid.

And it is here at this most important point that the fundamental error of our system is found.

In most English public schools senior mathematics is in the hands of men well qualified to fill their position. The mathematical staff of such a school comprises seldom less than four and in the larger schools frequently twice or thrice that number. The senior master and his first assistant are usually men of ability with first-class qualifications. The remainder may be and normally are men who have not passed beyond the standard required for a pass degree at an English university. Frequently they do not devote their time solely to mathematics, but in addition are engaged in teaching science or the humanities.

The senior master usually divides most of the upper school mathematics, especially the scholarship work, between himself and his more able colleagues. Matriculation and lower school work will normally be relegated to the less qualified assistants.

If at all, it is in the lower school that fundamental principles and logical methods are imbibed ; and it is here that men of conspicuous ability are required.

For the whole value of mathematics as an educational instrument lies in the logical development of the subject and in the habit of continued vigorous thought required to comprehend each successive step in long trains of reasoning.

It is the essential logicality of the science which must be instilled. Here is no place for slipshod methods or partial comprehension. Fundamental axioms must be clearly understood and on these must be built, step by step in logical sequence, the whole vast superstructure.

No man can teach clearly unless he understands the finest details of his subject. Mathematics is not a science of which even the elementary principles can be grasped by one who has not passed long years in patient study. Were he Newton himself, there is no royal road to mathematics. It is not until an advanced state of study and thought is reached that fundamental principles and assumptions come up for revision, that axioms and hypotheses can be placed upon a sound basis of mathematical argument. Until such knowledge is attained no teacher can clearly appreciate the finer points of elementary mathematics. And it is just these finer points which crop up for consideration in the class room. It is a poor class of twenty or thirty youngsters that does not produce one boy of sufficient intelligence and industry to see through the arguments of sophistry. Indeed the whole essence of mathematics is to take nothing on trust, but to search and search again for any suspicion of error.

Here, then, we must look for reform. We must have no dilettante mathematicians on our staffs. Much has been done in this direction since the introduction of specialisation into the scholastic world. Much remains to be done.

Passing to the consideration of what branches of mathematics should be comprised within the school curriculum, and how far each separate branch should be pursued, we are at once surrounded with difficulties.

In ordinary educational establishments as opposed to technical schools there has of late years been noticeable a marked tendency to regard the teaching of mathematics from a utilitarian rather than a purely educational point of view. The demands of industry and of commerce will make themselves felt here as in all other walks of life. It falls to the lot of our headmasters to examine this problem carefully, to ensure that future welfare is not sacrificed in a short-sighted policy of immediate return for present outlay.

Since the Industrial Revolution the population of England has been devoted almost exclusively to manufactures of every description. For a century now the industrialisation of England has proceeded apace; every branch of engineering expands from year to year; more and more youths on leaving school take to engineering as their life's work.

Engineering itself as a trade has long since passed from the experimental stage when machinery was an innovation and its working principles little understood. To-day it has attained to the dignity

of a science. The theory of modern engineering is highly abstract in nature and forms in itself the study of a lifetime. To understand its principles the whole range of the natural and mathematical sciences must be studied with care, sifted and collated.

Gone are the days when engineering practice could in a great degree disregard all theoretical considerations. Modern plant and machinery, its huge size and enormous speeds alike demand the most careful theoretical considerations; the vast structure of modern architecture, giant bridges and mammoth liners all require that no smallest detail shall be left to chance. All must be calculated with exact knowledge beforehand if disaster and untold expense are to be avoided.

The natural consequence of these changed conditions is to demand in all who propose to themselves engineering as a profession mathematical and scientific attainments of no mean order. The urgency of this demand increases as the years go by; our educational institutions are looked to and youths far advanced in mathematical knowledge are required from them.

The extensive application of the calculus, of trigonometry and of statics and dynamics to engineering practice require considerable proficiency in these branches of mathematics in all intending engineers. The question is as to how far these subjects should be taught at school, and at what age a boy should begin his study of them.

Each of these subjects requires a thorough and complete knowledge of arithmetic, of elementary algebra and geometry before they can be studied with profit and understanding. It is idle waste of time to try to teach a boy the meaning of a differential coefficient before he has yet learned to solve a quadratic equation, or to instruct him in the elements of trigonometry while he is yet ignorant of Pythagoras' Theorem. Yet this is the deplorable state of affairs in not a few schools of professedly engineering character.

Until the appearance of this unfortunate demand for the higher mathematics, it was usually considered sufficient for a boy to study arithmetic, algebra and geometry until the age of perhaps sixteen or seventeen. By this age the average boy may be expected to be thoroughly grounded in each of these subjects; to have reached what is termed matriculation standard. In very few schools is a boy allowed to specialise until he has passed the School Certificate or its equivalent examination. Until he can do so, the time which he can afford to devote to mathematics is necessarily very limited; usually it will not amount to more than one hour a day in class and perhaps an additional hour every other day in preparation. With so very little time available, it is quite impossible for a boy to study with any degree of success so diverse a range of subjects as arithmetic, algebra, geometry, trigonometry, calculus and mechanics. By the time he has reached the age of sixteen or seventeen he must be well advanced in the study of the first three, and it is hoped that he may be instructed in the elements of the last three. Perhaps one boy in thirty may be capable of so extensive

a course of study, but even he will suffer through learning the elements of calculus and trigonometry before he has completely mastered the groundwork of algebra and geometry.

The advocates of this extended syllabus will usually advance the argument that the calculus and trigonometry give added zest to the study of mathematics, while in mechanics we have a practical and utilitarian application of what has hitherto appeared to the student as of purely theoretical interest, if any.

It may be granted that an early initiation into the mysteries of higher mathematics may by the novelty of the subject give a fleeting stimulus to the young mind. Yet a mind which is so constituted as to lose interest in elementary subjects through long acquaintance and years of toil, will equally rapidly lose interest in any new subjects; subjects which furthermore are immeasurably more complex and difficult of comprehension.

The inevitable consequence of so ill-considered a course of instruction is that by the time the age of seventeen is reached the youth has a slipshod and superficial knowledge of many subjects, with no real knowledge and clear comprehension of any one branch of mathematics. During the past five years so much ground will have been covered that his attention will have been necessarily directed to the mechanical means of arriving at the answer rather than to a careful study of methods. So that he reaches the answer, he will rest content with his teacher's assurance that the method is correct and that each step can be demonstrated infallibly. To take a specific instance, almost any boy who has any knowledge of the calculus will know how to obtain a maximum or minimum value of any simple polynomial function. But very few will be able to advance reasons for their habit of equating to zero the first derived function or to explain when this gives a maximum value and when a minimum value. Such knowledge is to be obtained only by one who has covered his ground slowly and with care.

Finally, as the crowning folly of this system the boy, while yet struggling with the ill-digested food which has been rammed into his overburdened mind, is taught to apply his results to simple engineering problems. His ruin is complete. All pretence of the study or understanding of methods is here dropped. The engineer is interested solely in the answer; methods are beyond his consideration. The world over he works by rule of thumb, and his problems are peculiarly susceptible to such treatment. From the nature of his problems, much of his data is obscure. At the best, his answers are but approximations, which he recognises as such by the introduction of his famous Factor of Safety.

But that the growing mathematician should be taught such methods early, or that his young mind should be trained to such slovenly habits of thought, is to nullify the most valuable educative property of a mathematical training, the sustained habit of exact thought, and to ruin the foundations of all future knowledge.

From the foregoing considerations it appears that the course of study which will be pursued with most advantage by the average

boy is that of which the scope is limited to a thorough course of instruction in arithmetic, algebra and geometry until the age of sixteen or seventeen is reached.

Having advanced so far, he will be in a position to embark on a course of more advanced instruction, confident in his ability to make rapid and solid progress.

Grave dangers beset the path of youth wherever this tendency towards a too comprehensive syllabus prevails. We shall have to fear, not that the boy will learn too little at school, but that he will learn too much. An overgrown mind is just as injurious to sound health as an overgrown body, and if our schools become forcing houses for the production of what from the constitution of the human mind is normally the fruit of many years of assiduous industry, we may expect that this fruit will be correspondingly delicate and ill-adapted to withstand the wear and tear of after life.

A. N. HICKLING.

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862. I lodged in the Rue des Mathurins with two studious men, Lavirote and the Abbé de Prades: the latter was occupied in translating the theology of Huet, and the other the physics of Mackhlorin, a pupil of Newton.—*Memoirs of Marmontel*, 85. [Per Mr. F. Meixner.]

863. I was content and happy when in d'Alembert's little room at the house of his excellent glazier's wife, dining frugally alone with him, I listened to him—after he had been working on advanced geometry all the morning—speak as a man of letters, full of discernment, wit and knowledge; or on ethical questions, displaying the wisdom of a mature mind and the playfulness of a young unconstrained soul, he would survey the world with the eye of a Democritus and make me laugh at its stupidity and arrogance.—*Memoirs of Marmontel*, 148. [Per Mr. F. Meixner.]

864. Dans le calcul, il fait toujours choisir les notations les plus avantageuses; soit pour aider la mémoire, soit pour abréger les éliminations. Autant il serait ridicule de faire consister dans de simples conventions, la majeure partie des Mathématiques, autant il serait exagéré de les en bannir entièrement. Elles détruisent quelquefois l'aridité du calcul, et l'on pourrait dire que la notation est à l'Analyse, ce que l'arrangement et le choix des mots est à la clarté du style. Avec son secours, les calculs les plus longs deviennent une somme de transformations semblables, et il ne faut plus pour achever le tout que la patience de répéter la partie principale. C'est avec de pareilles inventions que l'Analyse algébrique, si pénible dans son origine, est parvenue au point de donner des leçons de simplicité à la Géométrie même.—Lamé, *Examen des différentes méthodes employées pour résoudre les problèmes de Géometrie*, § 18, p. 26. (1818; facsimile reprint.) [Per Prof. E. H. Neville.]

865. It was thought for a while that this might be due to a want of convergence of the infinite determinant, but some additional computations have led me to believe that the convergence, though slow, is sufficient, and in any case there is not the slightest hope that the particular terms, to which the negative root is chiefly due, will be cancelled out by terms of higher order now neglected. The true explanation must be sought in the mathematics of the method. Mathematics, like justice, is blind, and does not know what it does. Or, rather, it does the rigorously correct thing, even if asked to have consideration to circumstances. Mathematics can have no considerations. Also, of course, it can make no mistakes.—Prof. de Sitter, *Monthly Notices of the Royal Astronomical Society*, xci, 7, (1931) 715. George Darwin Lecture, on Jupiter's Galilean Satellites. [Per Prof. E. H. Neville.]

MATHEMATICAL NOTES.

1031. *A difficult converse.*

The difficulty of proving that if the bisectors of two angles of a triangle are equal, then the triangle is isosceles is well known ; three fairly simple proofs are given in this note.

(1) By Mr. M. J. Newell.

In the figure, BY and CZ are equal ; we have to prove that AB and AC are equal.

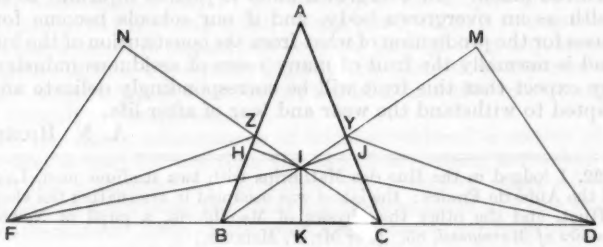

FIG. 1.

Produce BC both ways to D and F, so that $BF=AC$, $CD=AB$. Let fall perpendiculars FN, DM on the bisectors and draw IK perpendicular to BC. Then since I is the in-centre, we know that $AC+BK$, $AB+CK$ are each half the perimeter of ABC ; since $BF=AC$, $CD=AB$, we have $FK=KD$ and thus IFD is isosceles. Hence

$$IF=ID,\ \angle IFK=\angle IDK. \qquad \text{.....(i)}$$

The triangles FZB, AZC are equal in area since their bases FB, AC are equal by construction and their altitudes are equal since Z is on the bisector of the angle C. To each add the triangle BZC. Then FZC, ABC are equal in area. Similarly DYB, ABC are equal in area. Hence FZC, DYB are equal in area. But the bases CZ, BY of these triangles are equal by hypothesis, and hence their altitudes FN, DM are equal. Combining this with the results (i) above, it follows that FNI, DMI are congruent, and so the angles FIN, DIM are equal. By Euclid I. 32, we have

$$\tfrac{1}{2}\angle C+\angle IFC=\tfrac{1}{2}\angle B+\angle IDB,$$

whence by using (i) we have $\angle B=\angle C$.

(2) The main lines of the following proof are due to Mr. J. H. Doughty and were suggested to him by some remarks on this topic which he discovered in the *Lady's and Gentleman's Diary*, 1859, pp. 87-8, and 1860, pp. 84-6. The present improved form of Mr. Doughty's proof is due to Mr. J. Travers.

As in Fig. 1, the bisectors BY, CZ are equal, so that it is enough to prove that the triangles ABY, ACZ are congruent. We have therefore to prove the following theorem :

Two triangles are congruent if their bases, vertical angles and the bisectors of their vertical angles are equal.

Place the triangles in the circumcircle on their common base BC.

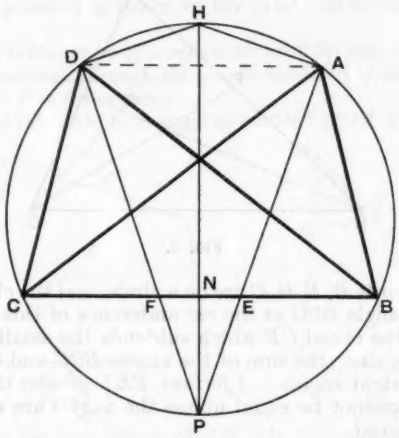

FIG. 2.

The bisectors DF, AE of the vertical angles will meet the circumcircle in P, the mid-point of the arc BC. Draw the diameter PNH. Since $FNHD$, $NEAH$ are cyclic quadrilaterals

$$PF \cdot PD = PN \cdot PH = PE \cdot PA.$$

Hence $DFEA$ is a cyclic quadrilateral, and since by hypothesis DF, AE are equal, it follows that DA and BC are parallel, whence

$$DC = AB.$$

Thus the congruence of the triangles ABC, DBC is established, and the result is proved.

(3) A very elegant proof was found by Mr. Doughty in the *Lady's and Gentleman's Diary* for 1860, p. 86; it is there attributed to the Reverend Wm. Mason, M.A., Vicar of Normanton, Yorkshire. The proof depends on a lemma tacitly assumed in the *Diary*, no mention being there made of the essential proviso attached to the lemma.

Lemma. *Of two unequal angles at the circumference of a circle, the greater will subtend the greater chord, provided that the sum of the two angles is less than two right angles.*

Proceeding to the main theorem, let ABC be a triangle in which the bisectors BE, CF of the base angles are equal. Suppose that the base angles ABC, ACB are not equal; then let ACB be the greater angle.

We have $\angle FCE > \angle FBE$.

Draw CG so that $\angle FCG = \angle FBE$.

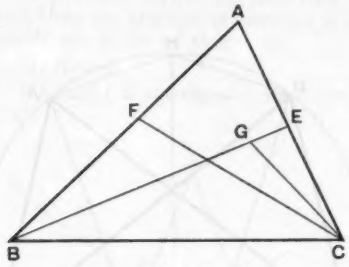

FIG. 3.

Then the points B, F, G, C are on a circle, and the chord BG which subtends the angle BCG at the circumference of this circle must be greater than the chord CF which subtends the smaller angle CBF, by the lemma, since the sum of the angles BCG and CBF is clearly less than two right angles. *A fortiori*, BE is greater than CF. Thus the bisectors cannot be equal unless the angles are equal, and the theorem is proved.

1032. *On Note* 1029.

On p. 133 of the current volume (xvi) of the *Gazette*, Mr. H. J. Curnow obtains the primitive of the equation

$$9xy^2\frac{d^2y}{dx^2}=2.$$

The primitive, in the form

$$\frac{\{y(x+Ay)\}^{\frac{1}{2}}}{Ax}-\frac{1}{2A^{\frac{3}{2}}}\log\frac{(x+Ay)^{\frac{1}{2}}+(Ay)^{\frac{1}{2}}}{(x+Ay)^{\frac{1}{2}}-(Ay)^{\frac{1}{2}}}=\frac{2}{3}\left(B-\frac{1}{x}\right),$$

is given (p. 221) in the second edition (1923) of my *Solutions to the Examples*. . . .

The result first appeared in a paper by Mr. William Brash, "Two general theorems in the differential calculus", *Proc. Edin. Math. Soc.*, vol. xxx (1911-12), pp. 107-116—a reference for which I am indebted to Professor H. Levy, of the Imperial College of Science and Technology.

6th May, 1932. A. R. FORSYTH.

1033. *On Note* 1006.

In a recent note * Mr. Forder has shown that the use made of a hyperboloid to prove a theorem on the elementary geometry of a tetrahedron, in Coolidge's *Circle and Sphere* (p. 235), is quite

* *Math. Gazette* XV (1929-31), 470.

unnecessary. Coolidge's general theorem can be deduced, as Mr. Forder states, in an elementary fashion from the following theorem.

$ABCD$ is a tetrahedron, circumcentre O, centroid G. The point P in OG produced such that $OG = GP$ is called the associate centre of $ABCD$. The point Q is taken in OG produced so that $OG = 3GQ$. Then :

The point Q is the centre of a sphere through the centroids of the faces of the tetrahedron and through the points one-third of the way from the associate centre P to the vertices.

The point of this note is to suggest another proof of this result.

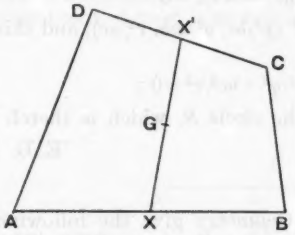

FIG. 1.

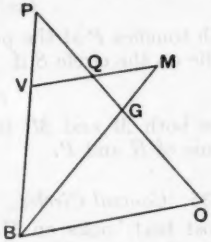

FIG. 2.

If X, X' are the mid-points of AB, CD, the plane at X perpendicular to AB, and the plane from X' perpendicular to AB are parallel, and G, being the mid-point of XX', is half-way between these planes. The first plane passes through O, hence the second passes through P, from the definition of P. Similarly for the other five such planes. Now, if M is the centroid of ACD, $BG = 3GM$; hence we have $QM = \frac{1}{3}OB = \frac{1}{3}R$, where R is the circum-radius.

Also if $PV = \frac{1}{3}PB$, $QV = \frac{1}{3}OB = \frac{1}{3}R$.

Hence the sphere centre Q and radius $\frac{1}{3}R$ passes through M, V and the similar points.

A. ROBSON.

1034. *Trivia.*

1. I have never seen in any text-book dealing with plane co-ordinate geometry this aspect of the circumcircle of a triangle. If P be any point on the circle ABC and D, E, F and N the projections of P on BC, CA, AB and the tangent to the circle ABC at A respectively, then

$$PE \,.\, PF = PD \,.\, PN.$$

Then if $PE = y$, $PF = z$, $PD = -x$, $PN = (bx + cy)/a$,

and $$yz + x(bz + cy)/a = 0,$$

or $$ayz + bzx + cxy = 0.$$

2. If the point O on the circle ABC be the centre of a rectangular hyperbola H with respect to which the triangle ABC is self-conjugate and also the focus of a parabola P inscribed in the triangle, then the points of contact of the common tangents of H and P lie on S, the circumcircle of the triangle, that is, S is the F-conic of H and P.

If $l+m+n=0$, $H \equiv lx+my+nz=0$,

and $$P \equiv \sqrt{lax}+\sqrt{mby}+\sqrt{ncz}=0.$$

Let H meet

$$S \equiv ayz+bzx+cxy=0,$$

at the point M ; then if $p+q+r=0$, the trilinear ratios may be taken as (aqr, brp, cpq) and hence

$$la^2/p^2+mb^2/q^2+nc^2/r^2=0\ ;$$

the tangent to H at M will be

$$lax/p+mby/q+ncz/r=0,$$

which touches P at the point $M'(p^2/la, q^2/mb, r^2/nc)$, and this point will lie on the circle S if

$$la^2/p^2+mb^2/q^2+nc^2/r^2=0\ ;$$

hence both M and M' lie on the circle S, which is therefore the **F**-conic of H and P.

E. G. Hogg.

1035. *Co-axal Circles.*

Most text-books on Modern Geometry give the following well-known theorem : If through any point P secants PXY, $PX'Y'$ are drawn to two circles whose centres are C and C' respectively, and if PM is the perpendicular from P to the radical axis of the circles, then

$$PX.PY-PX'.PY'=2\,MP.CC'.$$

But none that I have seen point out that whether the circles do or do not intersect, the theorem is true if P is within or without both circles ; but that if one circle is within or overlaps the other and the point P is within one circle and without the other, then

$$PX.PY+PX'.PY'=2MP.CC'.$$

J. W. Stewart.

1036. *The Converse of Pythagoras' Theorem.*

The interest of the following note lies in the fact that a well-known theorem of elementary geometry, the "Converse of Pythagoras", is shown to have an exception.

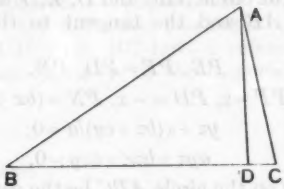

Fig. 1.

Given a triangle ABC in which $AB^2=AC^2+BC^2$, to prove that ACB is a right angle. Drop a perpendicular AD from A to BC.

$$AB^2 = AC^2 + BC^2 \pm 2BC \,.\, CD.$$

(Note that the assumption of this theorem, though contrary to the usually accepted order, is valid.)

But $AB^2 = AC^2 + BC^2$.

Hence $BC \,.\, CD = O$.

Thus either $BC = O$ or $CD = O$.

If $CD = 0$, the theorem is proved; but in the light of modern methods, ABC is still considered to be a triangle when $BC = 0$, that is, when B and C coincide: and in this case it is true that $AB^2 = AC^2 + BC^2$, but ACB is *not* a right angle.

Another point of view is as follows:

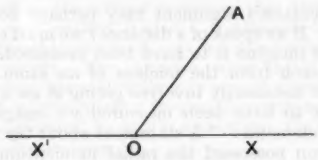

FIG. 2.

XOX' is an unlimited straight line, OA a finite straight line drawn from O. If any point P is taken at random on XX' a triangle AOP will be formed, and the point O should not be excluded from this statement. There are two positions along XX' at which P can be taken such that

$$OA^2 = OP^2 + PA^2.$$

In one of these positions OPA is a right angle. In the other, $\angle OPA = \angle X'OA$ or $\angle XOA$, so that there are *three* possible values for $\angle OPA$.

N. J. CHIGNELL.

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866. From *Conway Letters*, by M. H. Nicholson (Yale University Press. 1930), No. 168 (p. 265). Henry More to Robert Boyle. November 27, 1665.

"My quartan... was cured by this diatrion, fasting, sack, and mathematics; especially Oughtred's Clavis."

No. 284 (p. 479). Henry More to Dr. John Sharp. August 16, 1680.

"Mr. Newton has a singular Genius to Mathematicks, and I take him to be a good serious man. But he pronounces of the Seven Churches, not having yett read my Exposition of them, which I wrote by itself on purpose, concerning which another Mathematicall head, noted for that faculty in Cambridge, writ to me, that with Mathematicall evidence I had demonstrated that truth." [Per Mr. F. Puryer White.]

867. Stepping to the edge of the cut bank, they looked over. The precipitous slide of earth, almost as pale as snow at their feet, was gradually swallowed in the murk. The fact that they could not see the bottom of it made the leap appear doubly terrible. "Remember to let yourself go limp when you hit the dirt", he said. "Gravity will do the rest. I'll be there before you, because I'm heavier".—Hulbert Footner, *A Backwoods Princess*, p. 245.

REVIEWS.

The Physical Significance of the Quantum Theory. By F. A. LINDEMANN. Pp. vi, 148. 7s. 6d. 1932. (Oxford University Press)

This is a very interesting and original presentation of the ideas of the quantum theory. It is mostly non-mathematical, but is intermediate in difficulty between the avowedly popular expositions and the more advanced treatises. Lindemann takes the Principle of Indeterminacy as his starting point, and shows how the various quantum paradoxes can all be connected on to it. This undoubtedly goes straight to the root of the trouble, viz. that the separation of dynamics from kinematics, and the conception of space-time as a continuous background in which dynamical entities such as energy and momentum operate, is illegitimate beyond a certain point. The two conjugated variables, position and momentum, cannot rightly be abstracted from one another.

A glimpse of Lindemann's argument may perhaps be obtained from the following paraphrase. If we speak of a distance r we must either have measured it or at least we must imagine it to have been measured. Apply this to the distance r of an electron from the nucleus of an atom. The operation of measuring its position necessarily involves giving it an appreciable kick; so that if we imagine it to have been measured we imagine it to have been kicked. To quote Lindemann: "A statement about the radius vector would imply that the electron possessed the radial momentum which would have been acquired if the radius vector had been measured". Ingeniously he extends this to angular measures. In measuring an angle, you cannot make a mistake of more than 2π; so that in conceiving your variable as an angle, you must treat it as having been measured with an error not exceeding 2π, and as having the angular momentum averaging $h/2\pi$ which the process of measuring would impart. The exclusion principle is similarly treated. The positions of the electrons have to be defined sufficiently accurately for us to be able to count them. (Those who count sheep before going to sleep will recognise the justice of this requirement.) It turns out that the requisite measurements of position give the electrons momenta of different amounts, such that they occupy orbits of different quantum numbers. It is an application of the principle that we must be consistent even in our imaginations; if we say that there are three electrons present we imagine them to have been counted, and that in turn requires us to imagine sufficient distinctness of position, and so on.

I do not think that Lindemann's approach would be a satisfactory way of reaching the more mathematical developments of the quantum theory. He is not leading us in at the door; but he takes us to a window from which we get a peep, and a surprisingly ample peep, at the mysteries within. And his view certainly helps us to realise that after all there is a great deal of common sense even in the quantum theory. He refers several times to his having put forward the view in 1924 that space and time are unsuitable indefinables and the difficulties are caused by our inability to describe reality in other than spatio-temporal language; but I think this view was quite widely held before 1924; and certainly it had been explicitly stated. However that may be, he is to be congratulated on his success in the much more difficult task of developing its consequences. A. S. EDDINGTON.

A Treatise on Algebraic Plane Curves. By J. L. COOLIDGE. Pp. xxiv, 513. 30s. 1931. (Oxford; at the Clarendon Press)

This interesting volume is packed with valuable matter, but covers too wide a field to be complete in any part. Most of it is already accessible, not of course so conveniently in a single English volume. Nothing is explicitly

claimed as new; in some places the absence of any reference suggests a refreshing combination of originality and modesty. The bibliography is excellent.

It falls into four Books, which deal respectively with elementary theory, singular points, systems of points, and systems of curves. No attention at all is paid to curves of low degree, except in the last Book. Certain general classes, such as hyperelliptic curves, come in for ample treatment.

It is excellent in its free use of all other branches of mathematics that have a real bearing on the subject, from the fundamental theorem of algebra to the connectivity of a Riemann surface. The one branch that is curiously avoided is pure geometry in two dimensions. A disproportionate amount of space is devoted to these branches instead of to the main topic: it is almost a treatise on various subjects with applications to plane algebraic curves. These digressions fall between two stools: they are too sketchy for an expert and too condensed for anyone else.

In a disarming preface the author confesses that the real explanation of his choice of matter and method is that "it pleaseth him so to do". All through there are signs that the book was written for his satisfaction rather than for a reader's benefit. For example, what right has he to assume a public more patient than himself, willing to search twenty previous pages to discover what exactly is "the relation frequently mentioned in this chapter"? The style is further disfigured by slang, and worse, that has not always the excuse of brevity or effectiveness.

The algebraic work is heavy, and the page far too small for it; the printing is careful but monotonous. Elegance is made impossible by the use of rectangular cartesian coordinates requiring endless provisos about vertical tangents. In several other ways a little rearrangement would have helped; for example, an early statement of Noether's factorization theorem would have made it possible to give several other theorems more accurately.

In spite of faults, it is a book that every geometer will want not only to read but to possess. H. P. H.

Die Pseudosphäre und die nichteuklidische Geometrie. By FRIEDRICH SCHILLING. Pp. vi, 70. Geh. Rm. 3. Geb. Rm. 4. 1931. (Teubner)

This small book, which should be read with the same author's *Projektive und nichteuklidische Geometrie* (Leipzig, 1931), originated in a lecture delivered at the physical colloquium of the Technical High School, Danzig. Its object is to explain the concrete representation of hyperbolic geometry by the pseudosphere in the most elementary way, without using the difficult methods of the Calculus of Variations and the Theory of Surfaces.

Taking the asymptote of the tractrix (with constant r) as axis of Z and the tangent at the cusp as axis of Y, the property of the curve is expressed by the equation $Y dZ = -\sqrt{(r^2 - Y^2)}.\, dY$. For the surface of revolution, Y is replaced by ρ, and $X = \rho\cos\phi$, $Y = \rho\sin\phi$. A representation of the surface on the euclidean plane (ξ, η) is then determined by writing $\xi = r\phi$, $\eta = r^2/\rho$. The surface has constant negative curvature $-1/r^2$, and r is the space-constant of the hyperbolic plane. The element of arc on the surface

$$d\sigma = (dX^2 + dY^2 + dZ^2)^{\frac{1}{2}},$$

when expressed in terms of ξ, η is $d\sigma = r\,(d\xi^2 + d\eta^2)^{\frac{1}{2}}/\eta$. Bringing in now the complex variable $z = \xi + i\eta$ it is shown that the general homographic transformation $z' = (\alpha z + \beta)/(\gamma z + \delta)$ leaves $d\sigma$ invariant and is thus the representation of a non-euclidean motion. But this transformation of the (ξ, η)-plane changes circles into circles and, when α, β, γ, δ are *real* numbers, leaves $\eta = 0$ unaltered; it is also conformal or true to angle. Further, the meridians of the pseudosphere (which are geodesics) are represented by the parallel straight

lines $\xi=$const. which cut $\eta=0$ orthogonally, and these are transformed into circles cutting the ξ-axis orthogonally. Hence is obtained the well-known conformal representation of hyperbolic geometry in which straight lines are represented by circles cutting a fixed line orthogonally. The surface of the pseudosphere has a finite area and represents only a limited portion of the hyperbolic plane, viz. that bounded by two parallel lines and an arc of a limit-circle. This arc is represented by the cuspidal edge of the surface. (Somewhere Cayley tripped in a statement that the cuspidal edge is the representation of the absolute; this is represented by the axis of ξ, but the region of the plane lying between $\xi=0$ and $\xi=r$ is not represented at all on the pseudosphere.) The further developments are quite straightforward, and include the trigonometrical formulae, the different types of circles and simple motions, the area of the triangle, and the equations of geodesics on the pseudosphere obtained from the circles with centres on $\eta=0$ by transforming from ξ, η to ϕ, ρ. D. M. Y. S.

The New Graded Arithmetics. By WALTER SMITH. Books I-V. 5d. to 8d. each. Teacher's Books I-V. I-III, 1s. 6d. each. IV, V, 2s. each. 1924-1931. (Oxford University Press)

This series of books was completed last year by the publication of Book V; the earlier parts had already been adopted by the London County Council. They form a complete course in arithmetic for children between the ages of six and eleven. Thus, the earlier books carry on with the apparatus of the infants' school, tangible things of common use and coloured counters or discs; the numbers used increase gradually in size; the multiplication table is constructed with all due slowness; and some of the exercises are headed by pictures of goods sold in special kinds of shops with the prices marked. All this part of the course seems to have been constructed with special care. The notes for the teacher are good—though improvements could be made in a later edition. Division is described as either measuring or sharing; "goes into" is condemned, but it is proposed to replace the phrase by "measured" or "shared"—this is impossible, but one sees what is meant. Again, in one place we read that 6 and 8 are measures of 48; in another, that 100 is a measure of 50, 25, 20 and 10; this might confuse a child. The "shop" or "complementary" method of subtraction is preferred, but the traditional "equal additions" way is allowed. The later books do not call for so much comment, but by the time a child has come to the end of Book V he will have a competent knowledge of all the elementary processes with integers, decimals and fractions, much experience in calculating weights and measures and sums of money, and with all that a satisfactory grounding in such geometry as is necessary for understanding areas and volumes and the way to measure them. The whole series seems a very elaborate piece of apparatus for the little people for whom it was devised; but it has been so well thought out that there is little doubt that the teacher, the children and the books would between them produce an excellent result at the end of the five years during which they had been cooperating. T. M. A. C.

Everyday Arithmetic and Accounts. Second Year Course. By W. S. BEARD. Pp. 70, xviii. Paper, 1s. Cloth, 1s. 6d. 1931. (Oxford University Press)

This is the second part of a book of which the first has already been noticed. In a very attractive way it gives all the information and practice necessary for a preliminary understanding of such things as Interest, Investment, Bank Cheques, Bills of Exchange, Currency, the ways of registering accounts in book-keeping and the like. It should be found very useful in the Evening Institutes mentioned in the preface. T. M. A. C.

Oneven Toovervierkanten zijn geen Puzzles. By M. J. VAN DRIEL. Pp. 181, 97. No price given. 1931. (Versluys, Amsterdam)

An exhaustive monograph on magic squares of $(2n+1)^2$ cells. F. C. B.

Reeksen en Machten van den Scherpen Hoek. By P. J. VAN DEN BERG. Pp. 39. No price given. No date.

An elaborate treatment of ways, some very interesting, of splitting up arc tan 1 into various sums. Examples can be seen in the review of an earlier edition of the same work in the *Gazette*, volume xiv, page 248. F. C. B.

(1) **A New Algebra for Schools.** By C. V. DURELL. Part III. Pp. xiii, 198, xxxviii. Without appendix: with answers, 3s., without answers, 2s. 6d. With appendix: with answers, 3s. 6d., without answers, 3s. 1931. (Bell)

(2) **A School Algebra.** By T. C. BATTEN and M. W. BROWN. Part II. Pp. viii, 199-521. 4s. 6d. With answers, 5s. 1931. (John Murray)

(3) **A School Certificate Algebra.** By T. C. BATTEN and M. W. BROWN. Pp. ix, 460. 5s. With answers, 5s. 6d. 1931. (John Murray)

(1) This volume completes the course for additional mathematics in the School Certificate, and ably sustains the high standard set by its predecessor. Development of thought and idea is the characteristic note. The subject becomes increasingly abstract, and the treatment more and more theoretical, so that ultimately the "generalized arithmetic" of the first stages has disappeared, and the limits and infinite series of the higher mathematics have taken its place.

Negative and fractional indices form an introduction to logarithms, which receive a thoroughly "class-room" treatment. The logarithms themselves are obtained in the first place from a graph of 10^x, tables being employed only when the ideas are adequately appreciated. Numbers are written as powers of ten, and subsidiary calculations are done in side columns, but once the principles are mastered, the base is dispensed with, the whole of the work being done in parallel columns of number and logarithm. The logarithmic notation and its application to questions of inverse compound interest are reserved for a later chapter. The formula offers splendid scope for examples, and in this connection is exploited in wide and interesting variety.

Variation, direct and inverse, provide the material of a stimulating chapter. They are associated from the outset with their characteristic curves, of constant gradient and "constant area", the obtaining of the latter as the solution of the problem of rugs with equal areas but varying dimensions being particularly happy. The notion of irrationality is briefly mentioned, and the apparatus of surds established and applied to irrational equations.

Series form a milestone in the development of the subject. They are defined by means of a general term, its "formula" in terms of n becoming the statement of the law; a definite and significant advance in thought. A.P. and G.P. are comprehensively dealt with and H.P. introduced, while a geometrical treatment makes available the ideas of limit and limiting sum with the notation $n \to \infty$.

A special chapter emphasises the importance of Ratio and Proportion. These are exhaustively treated and reconciled with geometrical ideas of similarity. The graphical work is interesting. Linear and quadratic functions are discussed apart from tables of values, and elementary work in curve theory is made possible. A chapter is devoted to miscellaneous subjects, detached and undetermined coefficients, the Remainder Theorem with applications to cyclic expressions and factors. An ingenious development of the Binomial Theorem without the aid of Permutations and Combinations makes a delightful feature. Pascal's Triangle is obtained intuitively from simple

expansions and the coefficients in the general case deduced therefrom. Literal equations with a determinant solution of the simultaneous type, and elementary theory of quadratic equations and functions, complete the volume.

The examples, as one has learnt to expect from Mr. Durell, are more than adequate in number, interest and variety. They provide material for every type of pupil, and for indefinite revision.

(2) The authors in this volume complete an admirable text-book covering the School Certificate course both in elementary and advanced work. Manipulative work is firmly consolidated by means of harder fractions (latterly too much despised), square root and identities. The Remainder Theorem, with a functional proof, and applications, makes an early appearance. Equations are bound up with transformation of formulae and carried on to simultaneous linear and quadratic varieties. Graphical work becomes elementary analytical geometry. Equation and functionality are the keynotes, and the ideas of gradient and concavity are developed. The graphical solution of the quadratic is exhaustively discussed and the nature of the roots related. For indices is provided a refreshingly original treatment. They are shown to be particularly live inhabitants of the astronomical and biological universes and thereby take on an almost human interest. $\sqrt{2}$ is convincingly imprisoned between two converging sequences and surds become less elusive.

Logarithms follow the lines of Mr. Durell's work; a multiplicity of examples, mostly of drill variety, makes the progress very smooth. Series are restricted to A.P. and G.P., with applications to annuities and recurring decimals, and a masterly discussion of limiting sum.

The chapter on Ratio and Proportion introduces incommensurables, and contains a solution of simultaneous equations by cross multiplication without mention of the determinant notation. The treatment of variation is theoretical, and the areas of similar figures and volumes of similar solids serve as the seeds of dimensions. Graphical representation illuminates the development and provides, in the case of the "reciprocal function", an introduction to the asymptote and the notation $n \to \infty$.

Quadratic function and equation theory follows, with the obtaining of the eliminant of two quadratics. A paragraph is devoted to the cubic equation and its graphical solution by intersection. Permutations and combinations develop along traditional lines and lead up to the Binomial Theorem for a positive integral index, proved by induction.

The examples are pleasantly numerous and consistently satisfactory, with regular papers for revision. Historical notes, as in the case of the previous volume, add notably to the value and interest of the work. Messrs. Butler and Tanner, incidentally, are to be congratulated on a piece of superb printing. The type is excellent, the graphs in particular being marvels of clarity, and the book is a delight to read.

(3) The authors have aimed at a volume for the pupil whose algebraic ambition is merely matriculation. To this end they have made a selection of chapters from both parts of their *School Algebra* suitable for a two or three years' course, but presupposing some preliminary work. The atmosphere of the examination room is introduced by means of valuable revision papers culled from a variety of school certificates. E. L.

Progressive Trigonometry. By F. G. W. BROWN. Part I. Numerical Trigonometry and Mensuration. Pp. ix, 222. 3s. 6d. 1929. Part II. Plane Trigonometry. Pp. xi, 264. 4s. 1930. (Macmillan)

Written with the object of providing a joint course in mensuration and trigonometry, Part I covers the usual ground from the area of a rectangle to the treatment of frusta of pyramids and prisms, the necessary trigonometry being introduced as it is needed. It is doubtful, however, whether teachers will find themselves in agreement with the author's expressed desire of com-

bining mensuration and trigonometry into a coherent scheme. The usual trigonometric expressions for area owe their peculiar form to our bias towards the "unit square". But all of these expressions have not the potency of the single Euclidean theorem which states that areas are proportional to the second powers of corresponding sides—a theorem which implies that the area of any plane figure may be specified by $L \times L$. Such a simple fact as this, and the allied fact that the concept of an area does not depend on that of right angle, should not be obscured by a superabundance of formulae the main object of which is to help in computation.

Part II is better in conception and treatment than Part I. It completes the course commenced in Part I, but it may be used as a self-contained text-book on trigonometry. It contains more than is needed for any of the usual school examinations, and in addition includes examples on the Tucker, Lemoine and Brocard circles and related topics.

Apart from many errors of detail the treatment is, on the whole, good. There are, in both parts, many examples which have been taken from recent examination papers, the source often being indicated. Teachers should find these very useful. V. N.

Army Mathematics. Second edition. Part I. By C. JENNINGS. Pp. 64. Part II. By R. L. W. TOBUTT. Pp. vii, 248. Bound in one. 6s. 6d. 1932. (Oxford University Press; Humphrey Milford)

The first edition of this book, which caters for the needs of candidates for the Army Second and First-Class Certificates, was reviewed in the *Gazette* in January, 1930. It deals with such simple arithmetic and algebra as occurs in Army routine and contains plenty of examples and test papers with answers. The authors have produced a book in which candidates for the above examinations may put their trust. V. N.

Higher Course Geometry (being Parts IV and V of *A School Geometry*). By H. G. FORDER. Pp. x, 264. 6s. In separate parts: Part IV, 2s. 6d.; Part V, 4s. 1931. (Cambridge)

A School Geometry. By C. O. TUCKEY and P. W. C. HOLLOWELL. Pp. xvi, 340. 4s. 6d. 1931. (Christophers)

A Geometry for Advanced Division, Central and Secondary Schools. Part I. By J. W. M. GUNN. Pp. viii, 124. 2s. 1931. (Rivington)

A Companion to Elementary Geometry. By G. H. HAMILTON. Pp. vi, 87. 2s. 6d. 1931. (Blackie)

Higher Course Geometry is the continuation of Mr. Forder's *School Geometry* reviewed in these columns in July 1931. It is intended primarily for the mathematical specialists who are carrying on their work up to the standard of the Higher Certificate and University Scholarship examinations. The book starts with revision and extension of work on the triangle and circles connected with it. There follow chapters on directed lengths (theorems of Ceva and Menelaus), vector methods, homothetic figures and centres of similitude, and Solid Geometry. These form Part IV. Part V starts with Conical Projection and the Conic Sections, and goes on to Poles and Polars, Orthogonal and Co-axal Circles and Inversion. Three chapters at the end deal with miscellaneous topics, such as quadrilaterals in and round circles, Poncelet's theorem, theorems on four points and four lines, Gergonne's construction for the problem of Apollonius, Malfatti's problem, together with a discussion on maxima and minima and the geometry of displacements. Finally comes a note on the "Challenge theorems" set at the end of the earlier book. The ordinary properties of harmonic division are included and cross-ratio introduced in connection with conical projection, but the more formal treatment

of these is reserved for the promised book on Projective Geometry which is to round off and complete the author's design.

Readers of the earlier book will remember that Mr. Forder's idea was to write a school text-book in which the logical basis should be more clearly stated than is usual in modern books, and in which the standard of rigour should be gradually raised. In the present volume the greatest care is taken to show under what conditions theorems and constructions are valid, and to examine possible cases of failure. The logical necessity of existence theorems is stressed, and ideas such as congruency and similarity are rediscussed in the light of added experience. Mr. Forder does this sort of thing extremely well. Nothing, for instance, could be more lucid than his note on infinity on page 187.

To turn to details. In the first chapter we have the introduction of the "directed angle" or "Cross", a fruitful idea in the generalisation of theorems connected with angles. The work on vectors is well done, though it is perhaps a pity that the author did not use a special type (or the notation suggested on p. 29) to distinguish vector from scalar equations. The chapter on solid geometry is admirably clear and complete—particularly good is his treatment of skew lines and tetrahedra. The chapter on inversion is much fuller than in most books and includes work on inversion in orthogonal circles, antinverses and inversion in space. In this chapter Mr. Forder corrects "certain widespread errors of enunciation and demonstration". Prominent among these are two linked theorems, given by Casey and copied by other writers, dealing with relations between the radii and lengths of common tangents to two inverse circles. Mr. Forder contributed a note to the *Gazette* in October 1931 pointing out the errors and their correction. In noticing Casey's mistake, Mr. Forder has been anticipated by Mr. Roger A. Johnson, who refers to it in his *Modern Geometry* (Riverside Press, Cambridge, U.S.A., 1929); he does not, however, work out the results of the correction as fully as Mr. Forder.

There are one or two exceptions to the author's scrupulous care in the use of terms and phrases. He states Poncelet's theorem thus (p. 225): "If a variable triangle be inscribed in a circle of a co-axal system, and two of its side-lines touch two fixed circles of the system (which may be coincident), then the third side-line touches a fixed circle of the system". Now if any triangle ABC be inscribed in the first circle, AB touching the second and BC the third, all that can be said about AC is that it touches one of two fixed circles of the system. (For any position of A there are two possible positions for the tangent AB, etc.) In quoting this theorem this point is often overlooked. By "variable" Mr. Forder means to imply that the triangle moves continuously, the vertices moving round the first circle, the points of contact of the tangents moving continuously also round their circles. That this is his meaning becomes clear from the notes and footnote on p. 226, but the theorem might have been less ambiguously stated.

In the opinion of the writer, the least satisfactory part of the book is the geometrical conics in Chapter 5. The trouble about a first attack on this branch of the subject is the number of geometrical properties of the conic sections which are generally required to be known; moreover the links between them are not always easy to remember. The average pupil will probably require a more deliberate treatment of the subject than that here given and more easy examples. Mr. Forder states in his preface that the section contains "all the properties of the conic sections usually studied that can be readily treated by the methods of pure geometry. Proofs which are merely translations of analytical proofs into a less natural notation are excluded". But his selection of properties seems rather arbitrary. He does not attach much importance to the normal-properties. Thus the fact that the subnormal of a parabola is constant appears only as an exercise, while for a central conic the relations $SG = e.SP$ and $PF.PG = BC^2$ do not appear at all. Nor does $SP.HP = CD^2$. On the other hand, Newton's theorem on the

product of segments of two secants drawn to a conic from a point is surely one of those which is more easily treated by analysis. There are some interesting exercises in this chapter, though some of them are rather hard. Question 13 on page 152 runs as follows: "A try is scored at Rugby football at a point not between the goal-posts. Prove that the best point from which to kick the subsequent goal is on the arc of a certain Rectangular Hyperbola". It would have prevented possible misunderstanding if it had been explained that all considerations as to what happens in a vertical plane are to be left out of account, *i.e.* that what is required is the locus of a point in the horizontal plane so that the line joining the feet of the goal-posts subtends at it the maximum angle (subject to the other condition). The whole of this chapter would, however, form at any rate a valuable revision course for a pupil who had covered some of the ground before.

Putting aside these criticisms of minor points, there is no doubt but that Mr. Forder has written another very good book—though perhaps not quite the book he started out to write. For this is not merely an advanced school text-book. It has become interpenetrated by a treatise. Especially in the final chapters he carries the work beyond the present demands of scholarship examinations, and only the very exceptional pupil will have the ability or the time to complete the course as here laid down in his school days. But the value and interest of the book should not finish with the end of the school period. It will be useful to the university student, and teachers of mathematics will find it full of ideas and generally stimulating. The many mathematicians who like to do an occasional geometrical rider will delight especially in the miscellaneous examples at the end. The problem of Prince Rupert on p. 257 should wile away the time on quite a long railway journey. The appearance of the book is, at first, one of some congestion, although the type and general lay-out are up to the high standards of the C.U.P. In most geometrical text-books published in this country we are accustomed to find the road bountifully provided with sign-posts. Either the most important theorems come each at the top of a page or they are well spaced off from the rest of the text. Probably because he had so much material to get into comparatively small compass, Mr. Forder has omitted these spacings, and it looks at first as if it will be difficult to "see the wood for the trees". But when one gets down to the book the impression disappears, for the arrangement is really clear and the order natural. It does demand effort on the part of the reader, and an intelligent use of the index and of pencil markings in the margins; but it is none the worse for that.

Yet another "School Geometry"! The name of Mr. Tuckey as one of the authors is sufficient to indicate that it will be sound and with some individuality. The text follows the usual design, an informal course of drawing and discussion during which certain geometrical results are induced, followed by a formal course of deductions based on these induced results. Points of note are: (i) Proofs are given of the congruency theorems, based not on the method of super-position, but on an assumed "principle of congruence" that "any figure can be exactly reproduced anywhere". These proofs come early in the book, but a note suggests that they be omitted till a later, revision, course. (ii) The fourth case of similar triangles, corresponding to the "ambiguous case" generally omitted, is given here for the sake of completeness. Its use is, very occasionally, necessary in riders, and its presence is welcome. (iii) The proof of the theorem about the proportional division of two side-lines of a triangle by a line parallel to the base is based on areas, not, as usual, on the intercept theorem. Though this does not escape the logical difficulty of incommensurables, it pushes it further into the background. (iv) There is a novel arrangement of revision examples. There are over 600 of these and they are grouped as properties of the most commonly occurring geometrical figures. This plan should work very well. The average pupil of School

Certificate age requires any amount of revision, and this sort of exercise, directed towards a specific figure and exhausting its possibilities, will be certain to bear fruit. It is probably true to say that the common failure of such pupils faced with a set of miscellaneous riders is due to the lack of some previous half-way course such as this. The proofs of theorems are well set out with especial care taken as to references. Two footnotes reveal a pleasant sense of humour not usually exhibited in geometrical text-books: "Many like to refer to the conditions for congruence by using capital letters in their correct relative positions, viz. S.A.S.; A.A.S. or A.S.A.; S.S.S. The hint supplied by A.S.S. (even if it spelt backwards S.S.A.) must not be overlooked". And (with regard to the case of congruency, given right-angle, hypotenuse, side), "There is much hair-splitting at meetings of examiners as to what forms of words show that a candidate knows he is using this case of congruency. The candidate, *if he does know*, is strongly advised to be on the safe side. Abbreviations R.H.S."

In Mr. Gunn's little book more space than usual is devoted to scale-drawing, construction and use of scales, with applications to elementary surveying, and plan and elevation. The formal part of the work includes angles, congruence of triangles, parallelograms, equal intercepts, and proportional division. It seems rather early to introduce plan and elevation, and its treatment is necessarily rather sketchy at this stage, but it should be a useful help to "space-thinking" and suitable as a link with the handicraft work of the pupils for which the book is intended. The formal work is adequately set out and there is a fairly large number of easy exercises.

Can it be that the technique of the Children's Hour is beginning to have an effect on the style of writing text-books? *A Companion to Elementary Geometry* might have as a sub-title "Heart-to-heart talks with Uncle Alfred". The author explains in his preface that his object is, not to write another School Geometry, but to help students by instructing them how to learn from their text-books and by pointing out the common mistakes and pitfalls to which, in his experience, they are liable. Or, as he puts it later in the text, "these pages are only intended to be a kind of running commentary on the course of Geometry you have to pursue and to help things along by shedding a little more light here and there". There are few ideas in his 87 pages which are not used in class-work by every teacher of ordinary experience, though many things can be said which look rather foolish in cold print. A very backward pupil who felt he was making no progress, might get help from the book, if he were not irritated by its patronising style, but it is not possible to forecast for it any general utility. The course covered is, roughly, that of the London Matriculation. H. E. P.

The Theory of Spherical and Ellipsoidal Harmonics. By E. W. Hobson. Pp. xi, 500. 37s. 6d. net. 1931. (Cambridge University Press)

Professor Hobson needs no introduction as an author of mathematical works. Many of us became familiar at an early age with his *Treatise on Plane Trigonometry*, and his *Theory of Functions of a Real Variable* has for long been a standard work. The present volume on spherical and ellipsoidal harmonics will doubtless take its place, for many years to come, as the standard work of reference on the subject.

Professor Hobson has filled a definite need in producing this book. Anyone who has attempted to work with spherical harmonics will have realised the difficulty of finding the formulae he needs, particularly those concerning the functions of general order and degree. The difficulty has been increased by the fact that different authors have used different definitions for associated Legendre functions, and, if the present volume did nothing else, it would be valuable in standardising the functions in the same way that Professor Watson's *Theory of Bessel Functions* has standardised the different types of Bessel functions.

As the author remarks, the book is of a purely mathematical complexion, but the needs of the mathematical physicist have not been overlooked.

The book opens with an introductory chapter on the transformation of Laplace's equation. This is followed by a chapter on Legendre coefficients and Legendre's function of the second kind. Much of the material of this chapter is naturally to be found in many other books on analysis, but even at this early stage many results are given which are not readily accessible, and these make the chapter a most interesting one.

Chapter III is concerned with the associated Legendre functions with integral order and degree. This is a valuable introduction to the more general functions which are discussed later, and will no doubt be particularly useful to those investigators who are only concerned with the less general functions. The next chapter gives an account of the different types of spherical harmonics of integral degree, including an account of Maxwell's theory of the poles of spherical harmonics.

In Chapter V the general definitions of the associated Legendre functions of unrestricted degree, order and argument are given and developed. In working out this theory Professor Hobson had the choice of two methods. One of these was given by the author himself in a classical memoir published by the Royal Society in 1896, in which he used contour integrals taken round double circuits; and the other method was given by Barnes in an important memoir published in the *Quarterly Journal*. It was only natural for the author to choose his own method, but one feels that perhaps he might have included a short chapter illustrating Barnes' methods. He does mention the use that Barnes made of contour integrals involving gamma-functions, but the method is not developed at all. Doubtless the reason is that the chapter already runs to well over a hundred pages. This long chapter is followed by one giving asymptotic expansions of Legendre functions and various inequalities.

The style of Chapter VII is more reminiscent of that of the author's *Theory of Functions of a Real Variable*. Here a general theory of the convergence and summability of the series of Legendre and Laplace is worked out, and results are given similar to those of the well-known theory of Fourier series. This is followed by chapters on the addition theorems for Legendre functions of unrestricted degree, the zeros of the associated functions, and the ellipsoidal harmonics introduced by Lamé.

The book contains a great deal of original work by Professor Hobson himself. Some of this has already been published, but some is now apparently published for the first time. The printing is up to the usual high standard of the Cambridge University Press. The book will be a most valuable addition to the shelves of mathematicians, whether their bias is towards the pure or applied side, and we are much indebted to Professor Hobson for his new and monumental work.

W. N. B.

Lehrbuch der Funktionentheorie. Bd. II. Second edition. By L. BIEBERBACH. Pp. vi, 370. Rm. 20. 1931. (Teubner)

The "classical" part of the theory of functions of a complex variable is confined to the first volume of this work, and the second volume is devoted to the discussion of some of the leading problems of modern function-theory, such as conformal representation of simply and multiply connected domains, properties of "schlicht" functions and of functions regular and bounded in a circle, uniformisation, Picard's theorem, integral functions, and singularities of power series on the circle of convergence. The treatment of these topics includes the most recent developments, and the reader is brought within sight of the frontiers of knowledge. The concluding section, which is on a rather different footing, contains a brief account of some of the properties of the Riemann zeta-function and serves as an illustration of the general theory.

The service rendered by Professor Bieberbach in collecting this great wealth

of material into a volume of convenient size is too well recognised by readers of the first edition to need emphasising here. The second edition has been generally revised and brought up to date by the introduction of a little new matter. The principal additions are a generalised form of Julia's theorem (p. 112), Milloux' theorem (p. 134), an account of Bloch's work on Picard's theorem (p. 229), and Ahlfors's theorem on asymptotic values of integral functions of finite order (p. 277) in place of the less precise theorem of Carleman given in the first edition. For reasons not stated, the detailed discussion of the correspondence of boundaries in conformal representation given on pp. 55-64 of the first edition is suppressed in this edition. A few misprints and minor inaccuracies survive from the first edition.

Considerable prominence is given in the course of the book to the formulae of Jensen-Nevanlinna, and this is undoubtedly desirable in a general treatise of this kind. But it is open to question whether these formulae always make for simplicity in individual applications. In particular cases it seems preferable to take the maximum modulus principle as fundamental, and to introduce as required the auxiliary functions appropriate to special problems. For example, Landau's proof of Hadamard's fundamental theorem on integral functions seems much simpler and clearer than the proof given here (pp. 244-246). But this is perhaps a personal view which would not find general acceptance. It is in any case only a minor criticism of a book for which we have in general nothing but praise. A. E. I.

Applications of the Absolute Differential Calculus. By A. J. McConnell. Pp. xii, 318. 20s. 1931. (Blackie)

Cartesian Tensors. By H. Jeffreys. Pp. vii, 93. 5s. 1931. (Cambridge University Press)

The expressions "Absolute Differential Calculus" and "Tensor Calculus" are generally taken to have the same meaning, although strictly speaking the former is appropriate only to a special part of the latter, namely, the theory of the differentiation of tensors. The word *tensor* itself has undergone a complete change of meaning. It has been used in Anatomy since 1704 or earlier to denote a muscle that *stretches* or tightens some part. As a mathematical term it was introduced by Sir William Rowan Hamilton in his *Lectures on Quaternions* (1853) to denote the numerical magnitude of a vector, or what amounts to the same thing, the ratio in which a unit vector in the same direction must be *stretched* to transform it into the given vector (also in *Elements of Quaternions*, 1866). These uses of the word are closely connected with the Latin *tensus*, the past participle of *tendere*, to stretch, and are still the only ones recognised by the *Oxford English Dictionary*, but Hamilton's definition never became really firmly established among mathematicians and is now obsolete. Apparently the first to approach the modern definition, which is roughly that of a generalised vector, was the American physicist, J. W. Gibbs. In the latter portion of his *Elements of Vector Analysis*, privately printed in 1884 for the use of students but not published in the ordinary way, he applied his form of vector analysis to the study of elasticity. In discussing the six components required to specify a *pure strain* or homogeneous deformation, he at first used his *dyadics* (nearly equivalent to what we now call a tensor of the second order) and then changed his notation as follows: "A dyadic which is reducible to this form we shall call a *right tensor*. The displacement represented by a right tensor is called a *pure strain*". Gibbs' work remained very little known until many years had elapsed, but it may have influenced W. Voigt, a leading authority on Elasticity of Crystals, who in his work *Die fundamentalen physikalischen Eigenschaften der Krystalle* (Leipzig 1898) used the term "Tensortripel" in the same sense as Gibbs' "Right Tensor". The six components of the displacement were called the "Tensor

components", and the notation was extended to any magnitudes which for a new choice of coordinate axes were transformed by the same formulae as in the elasticity problem. The axes used were at first rectangular; when oblique axes are used, it becomes necessary to distinguish between covariant and contravariant vectors. The now well-known method of making this distinction clear by placing the indices below or above was due to G. Ricci (1889). It was Ricci and Levi-Civita who, following the pioneer work (with a different aim) by K. F. Gauss in 1827, G. F. B. Riemann in 1861, and E. B. Christoffel in 1869, have been chiefly responsible for the development of the new calculus. Most of their results were published between 1887 and 1901. It is strange that they had no single word for their central concept. M. Grossmann drew the attention of A. Einstein to their work, and Einstein and Grossmann developed it and applied it to relativity in 1913. In a later paper (*Die Grundlage der allgemeinen Relativitätstheorie*, 1916) Einstein, probably acquainted with Voigt's writings on elasticity, introduced the word tensor, which has always been used since in the absolute differential calculus. To Einstein also is due the summation convention, that whenever a literal index occurs twice in a term, the sum of such terms is to be taken for all possible values of the index. This is more than a mere abbreviation, for its use actually suggests what is generally the best way to continue an investigation. Einstein's work has made the achievements of Ricci and Levi-Civita known to the whole mathematical world instead of, as formerly, to a select few. The fact that the vanishing of all the components of a tensor in one system of coordinates necessarily implies a similar result in any other system seemed to show that such vanishing was the typical form of a physical theorem. But it has recently been found that some physical laws need more than tensors (in the ordinary sense of the word), for the linear wave equation obtained by P. A. M. Dirac in 1928 (an equation of fundamental importance in quantum theory and relativity) showed, in the words of C. G. Darwin, "that apparently something has slipped through the net". The consideration of Dirac's equation has led Sir Arthur S. Eddington (1928-31) to a calculus of ψ-tensors, which cannot be reached from the ordinary tensor calculus, but from which the ordinary tensor calculus can be reached. It is from Dirac's equation expressed in terms of this new calculus that Sir Arthur Eddington has recently obtained his striking calculations of the mass of the electron and proton.

After this somewhat lengthy general discussion, we turn to the details of the books under review. The strong points of Professor McConnell's treatment are comprehensiveness, lucidity, and a very large collection of examples. There is no need for any one reader to work them all, but if he should find himself in doubt as to the application of any point in the text he can always clear up the difficulty by choosing some of the examples that are given at short intervals. This feature of the book should make it very valuable to students. The treatment is divided into four parts. Part I (pp. 1-34) gives the algebraic part of the subject, including most of what is needed on the pure theory of tensors. Part II (pp. 35-129) gives the application to algebraic geometry in considerable detail, including some rather unexpected matter such as the theory of elementary divisors (pp. 97-102). Part III (pp. 130-217) starts with 26 pages on curvilinear coordinates and covariant differentiation, and then gives a most interesting discussion of the differential geometry of curves and surfaces. Part IV (pp. 218-302) deals with particle and rigid dynamics, electricity and magnetism, elasticity and the special theory of relativity. It might have been better to devote less space to Part II and much more to Part IV, which includes more subjects than can be adequately treated in a short compass. Finally we have a chapter (pp. 303-313) on orthogonal curvilinear coordinates, a short bibliography, and a very full index. It should have been mentioned that at the beginning of the book is an unusually clear and detailed table of contents. We regret the lack of historical references.

Dr. Jeffreys' object is the laudable one of bringing the tensor calculus in its simplest form to the notice of students who are not sufficiently advanced to deal with either its theory or application in the most general way. He uses only rectangular Cartesian axes, so that "the distinction between covariant and contravariant vectors disappears, and with it the terms arising from curvature of the surfaces of reference". This certainly removes one of the difficulties of the subject, but some may consider that in disregarding the scheme of upper and lower indices we neglect one of the attractive features and lay up trouble in store for those who proceed further in their reading. The 93 pages are divided into nine short chapters dealing in an elementary manner with Cartesian tensors, geometrical applications (mostly the straight line and plane), particle dynamics, rigid dynamics, statics, continuous systems, isotropic tensors, elasticity and hydrodynamics, concluding with a brief index. Most of the chapters contain a few examples. With the limitation to Cartesian coordinates there is no possibility of dealing with relativity, and indeed the formidable character of the tensor formulae required for such applications would be foreign to Dr. Jeffreys' object, which is to illustrate the convenience of tensor methods in such branches of applied mathematics as are usually studied by undergraduates. H. T. H. PIAGGIO.

Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen. By K. HAYASHI. Pp. 125. Rm. 26. 1930. (Springer)

This is an extensive collection of tables to a large number of significant figures of the functions named in the title. Sixty pages are devoted to Bessel functions, and these contain a reprint of Meissel's great table with the range extended from 15·50 to 25·10 in addition to numerous tables of functions whose order is either an integer or half of an odd integer; an interesting feature is the inclusion of a number of graphs which illustrate the behaviour of $J_n(x)$ as n increases for various fixed values of x.

In the second set of tables complete elliptic integrals and various combinations of theta-functions with argument zero are tabulated as functions of k^2 with interval 0·001, the results being given to eight or ten significant figures.

The third set of tables gives the Legendre polynomials $P_n(\cos\theta)$ as functions of $\cos\theta$ with interval 0·01 and as functions of θ with interval 1°. These tables have been newly computed, the author being apparently ignorant of the existence of Tallqvist's tables of $P_n(x)$.

A supplement gives numerous fundamental formulae concerning Bessel functions of both kinds and also an account of the arithmetic-geometric mean. The author follows the example of various writers in using a special suffix to denote Bessel functions of non-integral order, but, by a curious metamorphosis, the ν normally used has become v.

A very useful feature is the inclusion of a large number of auxiliary tables (such as tables of powers of π) which were used in constructing the main tables; by means of these the reader will find the task of extending the main tables substantially lightened.

It is a regrettable necessity to feel compelled to make adverse comments on the accuracy of these tables. I am not a professional computer and I have not made a systematic check of the tables, in consequence of lack of facilities for carrying out such a check in a speedy manner; but, in the course of half an hour spent in running my eye down the table of $J_0(x)$, I detected two errors in the fifth significant figures (the third figures entered) of the entries for 19·51 and 21·95 respectively. Professor J. W. Campbell has stated, *Bulletin American Math. Soc.* 34 (1928), pp. 528-530, that, as a result of checking a small fraction (about a twentieth) of Hayashi's previous work *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen*, he detected 55 errors; from the ease with which I detected errors in the volume at present

under review, it is a natural inference that Professor Campbell's criticisms have not been successful in stimulating the author to attain a higher standard of accuracy. G. N. W.

Siebenstellige trigonometrische Tafel. By H. BRANDENBURG. Second edition. Pp. xxviii, 340. Rm. 36. 1931. (A. Lorentz, Leipzig)

In preparing the first edition of this work the author was evidently in ignorance of the existence of the 15-decimal tables of Andoyer, nor does he make any reference to these in the second edition, which now appears with the errors of the first corrected. No doubt the offer of a reward for each error discovered in the first edition resulted in a fairly complete comparison with Andoyer's tables, so that the second edition should be reliable.

In addition to the main table of natural sines, cosines, tangents and cotangents for every ten seconds of arc from 0° to 90°, with first differences, there are two subsidiary tables, namely a table of cotangents for every second from 0° to 6° to seven decimals, and a table of sines and tangents for every 10″ from 0° to 1° with seven significant figures. This last was provided to give the same seven significant figure accuracy in the early part of the table as that which is obtainable by the use of logarithmic sines, more especially in calculations involving cosecants of small angles.

The 31 figure approximation to e on page 336 is in error in the last 11 figures, which should read 60 287 471 353. This curious mistake is due to the omission from the summation of $1/n!$ for $n=21, 22, \ldots, 30$. Using continued fractions D. H. Lehmer in 1926 calculated e to 707 places, a calculation which is interesting in that it involved what is probably the longest division ever performed.

The printing of the book is clear, the binding is strong and the whole production is excellent. L. M. M.-T.

Mathematical Tables. Vol. I. Circular and Hyperbolic Functions, Exponential, Sine and Cosine Integrals. Factorial (Gamma) and Derived Functions. Integrals of Probability Integral. Prepared by the British Association Committee for the Calculation of Mathematical Tables. Pp. xxxvi, 72. 10s. 1931. (Office of the British Association, Burlington House, London)

Since the first report of the Committee for the Calculation of Mathematical Tables was published in 1873, many isolated tables have appeared in the annual reports of the British Association. The present volume is the beginning of a policy, which should be warmly welcomed, of collecting these scattered publications into volumes of mathematical tables with a sufficiently wide range of arguments to represent properly the functions tabulated and at the same time to provide proper facilities for interpolation. This policy, if consistently pursued, will entail the systematic extension of some tables and the provision of others not already calculated. In this volume interpolation is provided for by printing differences of even order with a view to the application of Everett's formula. In the case of functions whose values are only occasionally required this practice may be condoned on account of the obvious economy in printing. From the point of view of the user of tables, it would be a matter of regret if the omission of first differences ever came to be regarded as standard practice, which at present is fortunately not the case. A printed first difference saves one multiplication, provides a check on transcription and would have been welcome in such a generally useful table as that of sines and cosines with the natural numbers as argument.

The principal contents are $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, $\sinh \pi x$, $\cosh \pi x$, $\mathrm{Ei}(x)$, $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $\Gamma(x+1)$ which is called $x!$, $\int_0^x \log_{10} \Gamma(t+1)dt$,

$$\frac{d^n}{dx^n}\log\Gamma(x+1), \quad n=1, 2, 3, 4,$$

which are here called the di-, tri-, tetra-, and pentagamma functions, and a table of *Hh* functions defined by

$$Hh_0(x)=\int_x^\infty \exp(-\tfrac{1}{2}t^2)dt, \quad Hh_n(x)=\int_x^\infty Hh_{n-1}(t)dt, \quad (n=1, 2, \dots, 17).$$

This last table is interpolated by means of Taylor's series, for which purpose a table of

$$Hh_{-n}(x)=(-)^n\frac{d^n}{dx^n}Hh_0(x), \quad (n=1, 2, \dots, 7)$$

is provided.

The origin of the tables, their nature and interpolation is fully explained in the introduction and well illustrated with numerical examples. The volume is well printed and moderately priced.

The committee who produced this volume and their predecessors who laid the foundations are to be congratulated. L. M. M.-T.

Practical Mathematics. By L. Toft and A. D. D. McKay. Pp. vi, 594. 16s. 1931. (Pitman)

That the authors' view of what is meant by "Practical" mathematics differs considerably from that current a few years ago will be immediately evident upon realising that this is a volume in Pitman's "Engineering Degree" series. It was written with one dominant aim, that of adequately covering the requirements of the candidate who elects to take Mathematics as a subject at the B.Sc. (Engineering) examination of the University of London. For this it seems admirably suited, for the prescribed syllabus, interpreted in the light of questions already set, is closely followed, and the exercises at the ends of the chapters contain a large proportion of such examination questions.

Any criticism of the choice of subject-matter would be a criticism of the prescribed syllabus, and will not be attempted. As the authors state, this syllabus is wide, including both Pure and Applied Mathematics, and they acknowledge, with apparent regret, that to cover it adequately, considerations of space prevented them from straying outside its limits. Consequently we do not get quite so close to engineering problems as in Low's book, reviewed in the May *Gazette*. On the other hand, the omission of much elementary (and some advanced) matter enables a much more careful treatment to be given of such fundamental ideas as limits and convergency. In this respect it is good to note that explanations are always careful and accurate, and difficulties are not passed by unnoticed or hidden, but even when space will not permit of their complete resolution they are at least pointed out. Teachers should welcome the book, and the text should prove ample for the private student, especially as a large number of questions are worked through with generous explanations of the steps, and as complete answers to the numerous exercises, together with hints for the solution of the more difficult ones, are to be found at the end of the book. The student who has mastered the contents of this book should face his degree examination with equanimity and will have clear ideas upon mathematical fundamentals, together with considerable facility in working the mathematical machine. From another standpoint, however, one feels that his mathematics may still be more or less abstract, and that he will not have had much practice in reducing practical problems to such form that the mathematical machine may be set to work.

While the book is, like all Pitman's books, well got up and clearly printed, the set-up of mathematical formulae is occasionally unhappy. We would like to make one other criticism, namely, that if convention demands page headings, they could be made informative of the contents of the actual page, and

thus serve a useful purpose, instead of repeating throughout a chapter the topic which happens to come first in the chapter. W. G. B.

Philosophy and Modern Science. By H. T. DAVIS. Pp. xiv, 335. $3.50. 1931. (The Principia Press, Bloomington, Indiana)

The revolutionary changes which have been made during the last fifteen years in our ideas of the constitution of the physical universe have interested many people who have not had sufficient mathematical training to be able to read the memoirs and treatises in which these changes have been set forth. Even writers of detective stories make their characters discuss Einstein and Eddington, and possibly it will not be long before the q-numbers of Dirac become equally popular. The intelligent man is as anxious to understand the doctrines of the expanding universe and the uncertain electron as he is to understand the gold standard. The scientists have not overlooked this anxiety; a number of recent books deal in non-technical language with the problems and triumphs of modern mathematical physics and with the wider philosophic issues which are involved.

Professor Davis, known in this country mainly for his work on integral equations, has attempted to survey the whole field of modern physics, to relate seemingly divergent lines of philosophic thought, and to expound his subject without using mathematical symbolism. In addition, the preface tells us that "the book was originally undertaken to prove to a friend that there is much that is poetical in the story of modern science". Now it is clear that the author has set himself an enormously difficult task, and he could hardly expect to be successful at every point. The size of the book is small in relation to the field of its inquiry and so the aim "to set forth as simply as possible the basic postulates of physics" is sometimes defeated by the exigencies of space; and surely a great deal of the poetry is lost by the rejection—unfortunately necessary—of mathematical symbolism, which stands to physics as does metre to poetry. But if we are prepared to make considerable allowances for these almost unavoidable defects, then we can assert that Professor Davis has written a most interesting and useful book.

The first two hundred pages deal with the growth of modern theories of the universe, tracing them from Galileo and Newton through the various ether-theories to the Relativity concepts. The remaining portion deals with electron and quantum theory, including a chapter on Probability. The style is easy, degenerating only occasionally to the free-and-easy: true simplicity of style in expounding such difficult problems is given only to an Eddington or a Jeans, but Professor Davis is obscure only when he attempts to define some important concept—for example, Least Action—in four lines and no symbols.

One or two minor criticisms: the immortal *Alice* can be quoted too often, and the author's attempt to conjure with the rod of Carroll in introducing Alice to Socrates was rash; and what authority is there for the names of *MacCullah* and *Larmour*, versions which are obviously not due to the printers?

The general get-up of the book is admirable, and though the author's reach has somewhat exceeded his grasp, his work should be of considerable benefit to those who wish to know what the physicists really have been doing to the universe. T. A. A. B.

Abstracts of Dissertations approved for the Ph.D., M.Sc. and M.Litt. Degrees in the University of Cambridge for the academical year 1929-1930. Pp. 125. No price given. 1931. (Cambridge)

These abstracts are of interest in showing the scope of the research work which is being done by post-graduate students at Cambridge. There are abstracts of eight theses for the Ph.D. degree in the Faculty of Mathematics,

and these deal with such differing problems as the propagation of electric waves; the outer layers of the sun; the unitary field theories of gravitation and electricity; magnetism; parallaxes, absolute magnitudes and motions of certain stars; vortex motion; four-dimensional geometry; Fourier series; and the theory of measurable sets of points. T. A. A. B.

Problèmes d'Agrégation. (Mathématiques Élémentaires.) By J. DOLLON. Pp. 92. 15 fr. 1931. (Librairie Vuibert, Paris)

This is a collection of seventeen problems, set during the years 1905-1930, with solutions. One of them is algebraic, the rest are geometrical, and are solved mainly by purely geometrical methods. Each problem is multiple in form, taking perhaps a page of print for the statement of all its parts, and in the solutions M. Dollon has been careful to avoid doing too much for the reader. A Scholarship class might derive considerable benefit by working through the problems and comparing solutions with M. Dollon's, while a teacher would be unlucky if the book did not provide him with a few useful tips. T. A. A. B.

James Clerk Maxwell: a Commemoration volume, 1831-1931. Essays by J. J. THOMPSON, PLANCK, EINSTEIN, LARMOR, JEANS, GARNETT, FLEMING, LODGE, GLAZEBROOK, LAMB. Pp. 146. 6s. 1931. (Cambridge)

Addresses delivered at the Clerk Maxwell centenary are here collected into more permanent form. No reader of the *Gazette* needs to be told that the joint-authors of this volume are men speaking with authority. All that is here necessary is some slight indication of the scope of each contribution.

Sir J. J. Thompson's contribution, the longest in the book, is a short sketch of Maxwell's life, with special reference to the development of his electrical researches. Professor Planck deals with Maxwell's influence on theoretical physics in Germany, Professor Einstein with his influence on the conception of physical reality. Sir Joseph Larmor's essay, entitled "The Scientific Environment of Clerk Maxwell", relates the brilliant achievements of recent days to "a purely intellectual lead, flowing largely from the genius of Clerk Maxwell". Sir James Jeans writes on Maxwell's method, William Garnett on his laboratory, Sir Horace Lamb on his lecturing. Sir Ambrose Fleming and Sir R. T. Glazebrook both contribute memories of early work in the Cavendish Laboratory, while Sir Oliver Lodge fittingly writes on Maxwell and wireless telegraphy.

The essays in this volume—and, we would add as an inevitable companion, the Rede lecture of 1923 delivered in Cambridge by Lorentz—are tributes worthy of the man whose work inspired them. T. A. A. B.

Four-figure Mathematical Tables. Arranged by J. DOUGALL. Pp. 32. 1s. 1931. (Blackie)

These tables contain logarithms, anti-logarithms and co-logarithms, natural and logarithmic trigonometrical functions, and short tables of reciprocals, squares, square roots, cubes and cube roots, and exponential functions.

The co-logarithm of a number is the logarithm of its reciprocal, and the table of co-logarithms is for the purpose of dispensing with subtraction in logarithmic calculation. The beginner might easily get confused in assigning the correct integral parts to both logarithms and co-logarithms, and for this reason we hesitate to commend the idea for use at this stage. T. A. A. B.

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868. La veille, tout en buvant avec les rouliers, tout en fumant, tout en chantant des gaudrioles, il avait passé la soirée à observer l'étranger, le guettant comme un chat et l'étudiant comme un mathématicien.—Hugo, *Les Misérables*, II, 3. Chap. ix. [Per Mr. J. B. Bretherton.]

CORRESPONDENCE.

ANALYSIS OF REPLIES TO THE QUESTIONNAIRE ON THE TEACHING OF GEOMETRY.

To the Editor of the *Mathematical Gazette*.

330 BANBURY ROAD, OXFORD.

DEAR EDITOR,—I have had 190 replies to my two queries; among the replies being some welcome ones from overseas. I am very grateful to all my correspondents.

To take [II] first, that the angular points of congruent or similar triangles should be quoted in corresponding order, there was of course no dissentient voice though many ridiculed the expression "supreme importance". With regard to the placing of one set below the other, most welcomed it for similar triangles but were doubtful in cases of congruence. There was also doubt about "insisting". This I am sorry for, as it is an essential part of good style to be careful of "order". Examiners must condone many slovenly faults, but teachers need not.

However, the chief interest attaches to [I], that the fundamental theorem on areas is "Parallelograms on the same base and between the same parallels are equal in area".

151 are in favour: some most emphatically;
14 are indifferent, or have no decided views;
6 think it does violence to freedom of thought;
18 prefer the mensuration method of dealing with such equalities;
1 says he would like to scrap this theorem altogether.

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May I in further advocacy of this theorem, as standing in a class by itself, explain that accepting it does not tie a teacher to *start* with it, or to make no use of the mensuration methods when they are obviously advantageous. They can be deduced immediately afterwards. But this theorem asserts equality of areas, and proves it, as a matter of pure geometry, quite irrespective of how areas must be measured. It is as absurd to say that this theorem holds *because* each parallelogram equals a certain rectangle as it would be to say that two lumps of gold which balance are equal *because* the same number of sovereigns could be coined from each: they are equal *because they balance*, quite irrespective of other particulars. Well, my parallelograms are my lumps of gold! And there I leave the matter. I can prove they balance, without bringing in a sordid rectangle.

25 May, 1932. ALFRED LODGE.

A CHALLENGE.

To the Editor of the *Mathematical Gazette*.

SIR,—In the days of old when mathematicians understood one another, and tempered the wind to the shorn lamb, and their science was a Whetstone of Witte, it was customary for people such as Fermat and Pascal and even Leibniz or a Bernoulli to fling challenges to their friends in playful or serious mood; and progress in knowledge has notoriously resulted.

Perhaps the abeyance of this custom has not been an unmixed good. A fair test of this would be the performance of a modern scientist, trained to research, on an old-fashioned Cambridge problem paper, such as was set, for instance, by Stokes or Kelvin or Clerk Maxwell in days gone past.

As an illustration of the method of challenge, the present writer would be assisted by an answer to the following, which will doubtless be forthcoming —perhaps, indeed, more than one answer—as being well within the range of readers of your journal, and moreover of actual use as well as of abstract interest.

From a statistical table of duration of life the expectation in years of a person of given age can be read off. For example, a man of 75 may expect to have about 6 more years to live. How is the expectation to be deduced from the table, in years, that of two persons of the same age (i) *one of them will still be alive after that number of years, or* (ii) *both of them, or* (iii) *neither of them?*

Very faithfully,

WRANGLER.

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE Thirteenth Annual Meeting of the National Council of Teachers of Mathematics was held at the Hotel Raleigh, Washington, D.C. Three new standing committees were established: (1) Committee on Individual Differences; (2) Committee on Geometry; (3) Committee on Policy, which is to formulate a body of *first principles* that should govern the teaching of secondary mathematics. The Committee on Co-operation with Official Examining Boards was continued for another year, as well as the Committee on Handbook in Mathematics.

The National Council of Teachers of Mathematics, which had a very modest beginning in Cleveland, Ohio, February 1920, when approximately seventy-five charter members created the organization, has now grown to a membership of nearly six thousand, representing every state in the Union and many foreign countries.

ERRATA.

Vol. XVI., p. 122, eqn. (4). *For* $-q$ *read* $+q$.

p. 128, l. 1 up. *For* $C(z^n+z^{-n})=2C\cos 2u$
read $C(z^n+z^{-n})=2C\cos 2n\theta$.

p. 128. f.n. *For* $\frac{1}{2}\tan^4\phi$ *read* $\frac{1}{2}\tan^4\frac{1}{2}\phi$.

p. 129, l. 1. *For* $2u\theta$ *read* $2n\theta$.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. The names of those sending the questions will not be published.

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